GEOMETRIC ANALYSIS ON CONFIGURATION SPACES

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Abstract

The configuration space $\Upsilon(X)$ over X is the set of all point measures whose number of points is finite on any compact set. While various statistical physical aspects have been studied in relation to interacting particle systems over many decades, its geometric properties as infinite-dimensional metric measure spaces have been understood only relatively recently.

In this talk, we elucidate geometric aspects of $\Upsilon(X)$ *over possibly non-smooth* X with respect to the Wasserstein distance d_2 and a certain class of reference measures μ. We stress here that the standard metric measure theory does not work in $\Upsilon(X)$ due to the fact that the distance d_2 is not continuous (only lower semi-continuous), $\mathsf{d}_2 = +\infty$ on sets of positive measures, and μ is not doubling. These are typical phenomena in infinite-dimensional spaces.

We prove two key properties connecting metric measure geometry and Dirichlet form theory, that is, *the Rademacher property* and *the Sobolev-to-Lipschitz property* (the latter was conjectured by Röckner-Schied '99 *J. Funct. Anal.*). We then go on to identify two energy structures: one is the Cheeger energy $(\mathsf{Ch}, W^{1,2}(\Upsilon))$ based on Lipschitz algebra; the other is the classical Dirichlet form $(\mathcal{E}, \mathcal{F})$ based on the lifted gradient on cylinder functions.

Our result demonstrates that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is *geometric*, i.e., that it can be completely reconstructed by the metric measure structure $(\Upsilon(X), \mathsf{d}_2, \mu)$ via the Cheeger energy:

$$(\Upsilon(X), \mathsf{d}_2, \mu) \overset{\mathsf{Ch}}{\rightsquigarrow} (\mathsf{Ch}, W^{1,2}(\Upsilon)) \;=\; (\mathcal{E}, \mathcal{F}).$$

Furthermore, our result also shows the opposite implication that $(\Upsilon(X), \mathsf{d}_2, \mu)$ is *analytic*, i.e., that it can be reconstructed by the intrinsic distance $\mathsf{d}_{\mathcal{E}}$ of the Dirichlet form $(\mathcal{E}, \mathcal{F})$:

$$(\mathcal{E}, \mathcal{F}) \overset{\mathsf{d}_{\mathcal{E}}}{\rightsquigarrow} (\Upsilon(X), \mathsf{d}_{\mathcal{E}}, \mu) \;=\; (\Upsilon(X), \mathsf{d}_2, \mu).$$

Finally, we elaborate two applications of this fundamental result. Firstly, we show that $\Upsilon(X)$ satisfies the Evolution Variational Inequality $\mathrm{EVI}(K, \infty)$ when the underlying space X is $\mathrm{RCD}(K, N)$. This result implies the synthetic lower Ricci curvature bound of $\Upsilon(X)$ and also implies the coincidence of the gradient flow of the entropy with the heat flow on the Wasserstein space over $\Upsilon(X)$. Secondly, we prove the integral Varadhan short-time asymptotic of infinite particle systems with respect to the Wasserstein distance d_2.

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