On the trivial regime of mixed p-spin spherical Hamiltonians with external field

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Introduction

The problem of considering the minima of random fields has played an important role in recent developments in computer science and machine learning. For example, the minimization of evaluation functions generated from randomly extracted samples is a central problem in the study of artificial intelligence, and there is a need to develop algorithms that converge to the minimizers faster. Furthermore, geometrical properties such as the number of critical points and the topology of level sets in a random fields have been studied as the basis for such studies. In this talk, we discuss the number of critical points of random energy given by the spherical SK model with an external magnetic field.

Setting and Main results

Let $H_N(\sigma)$ for $\sigma \in \mathbb{R}^N$, be a centered Gaussian process with covariance

$$\mathbb{E}\left[H_N(\sigma)H_N(\sigma')\right] = N\xi\left(\sigma \cdot \sigma'\right), \tag{1}$$

where the function ξ is given by the series

$$\xi(x) = \sum_{p\geq 1} a_p\, x^p,$$

with $a_p \geq 0$, $\xi(1) = 1$, and an infinite radius of convergence. Further, let $h \geq 0$ and, for every $N \in \mathbb{N}$, let $\mathbf{u}_N \in \mathbb{R}^n$ be a vector with $\|u_N\| = 1$ and set

$$H_N^h(\sigma) = H_N(\sigma) + hN\mathbf{u}_N \cdot \sigma, \qquad \text{for } \sigma \in \mathbb{R}^N. \tag{2}$$

We are interested in the number of its critical points on the unit sphere S_{N-1} in $\mathbb{R}^N$:

$$\mathcal{N}_N = \{\sigma \in S_{N-1} : \nabla H_N^h(\sigma) = 0\}. \tag{3}$$

Here ∇H_N^h denotes the spherical gradient of H_N^h, that is the gradient restricted to the tangent space of S_{N-1}. We are further interested in the behaviour of the maximum of H_N^h and its global maximizer

$$\sigma_* = \mathrm{argmax}_{\sigma\in S_{N-1}} H_N^h(\sigma), \tag{4}$$

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We show that the behaviours of $\mathcal{N}_N$, σ^* and $H_N^h(\sigma_*)$ depend on the strength of the magnetic field h. The following value plays a key role

$$h_c = \sqrt{(\xi''(1) - \xi'(1))_+}. \tag{5}$$

定理 1. *If $h > h_c$, then*

$$\lim_{N\to\infty} \mathcal{N}_n = 2, \qquad \text{in probability and } L^1, \tag{6}$$

and the behaviour of H_N^h at its maximizer σ^* is given by

$$\lim_{N\to\infty} \frac{1}{N} H_N^h(\sigma^*) = \sqrt{\xi'(1) + h^2}, \tag{7}$$

$$\lim_{N\to\infty} \sigma_* \cdot \mathbf{u}_N = \frac{h}{\sqrt{\xi'(1) + h^2}}, \tag{8}$$

$$\lim_{N\to\infty} \frac{1}{N} \nabla_r H_N^h(\sigma^*) = \frac{\xi'(1) + \xi''(1) + h^2}{\sqrt{\xi'(1) + h^2}}, \tag{9}$$

$$\lim_{N\to\infty} \lambda_{\mathrm{Max}}(\nabla^2 H_N^h) = 2\sqrt{\xi''(1)} - \frac{2\xi''(1) + (h^2 - h_c^2)}{\sqrt{\xi''(1) + (h^2 - h_c^2)}}, \tag{10}$$

where $\nabla_r H_N^h$ and $\lambda_{\mathrm{Max}}(\nabla^2 H_N^h)$ denote the radial derivative and the largest eigenvalue of the spherical Hessian of H_n^h, respectively, and the stated convergences hold in probability.

定理 2. *If $h < h_c$, then the expected number of critical points grows exponentially,*

$$\lim_{N\to\infty} \frac{1}{N} \ln(\mathbb{E}[\mathcal{N}_N]) = \begin{cases} \frac{1}{2}(\frac{h^2}{h_c^2} - 1 - \log \frac{h^2}{h_c^2}), & \text{if } h/h_c \in [\sqrt{\frac{\xi'(1)}{\xi''(1)}}, 1), \\ \frac{1}{2} \log \frac{\xi'(1)}{\xi''(1)} - \frac{h^2}{2\xi'(1)}, & \text{if } h/h_c \in [0, \sqrt{\frac{\xi'(1)}{\xi''(1)}}). \end{cases} \tag{11}$$

In this talk, I will explain the background and motivation for the study of the number of critical points in random fields, and discuss the ideas used in the proof. This work is based on the joint work with David Belius, Jirí Černý and Marius Schmidt.