Skorohod SDEs with a class of non-Lipschitz coefficients and non-smooth domains*

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Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space with the usual condition. Let D be a domain of $\mathbb{R}^d$ and denote its closure by $\overline{D}$. In this talk, we are concerned with the following Skorohod SDE:

$$\begin{cases} dX(t) = \sigma(t,\cdot,X(t))\,dB(t) + b(t,\cdot,X(t))\,dt + d\Phi_X(t), \quad t \geq 0, \\ X(0) \in \overline{D}. \end{cases} \tag{1}$$

Here, $B = \{B(t)\}_{t\geq 0}$ is a d-dimensional Brownian motion and $\Phi_X = \{\Phi_X(t)\}_{t\geq 0}$ denotes a reflection term, which is an unknown continuous function of bounded variation. Coefficients $\sigma\colon [0,\infty)\times\Omega\times\overline{D} \to \mathbb{R}^d\otimes\mathbb{R}^d$ and $b\colon [0,\infty)\times\Omega\times\overline{D}\to\mathbb{R}^d$ are measurable functions. We assume moreover that $[0,\infty)\times\Omega \ni (t,\omega)\mapsto \sigma(t,\omega,x)$ and $[0,\infty)\times\Omega\ni(t,\omega)\mapsto b(t,\omega,x)$ are progressively measurable for any fixed $x\in\overline{D}$.

If D satisfies the conditions **(A)** and **(B)** below, and σ and b depend only on the space variables and are Lipschitz continuous on $\overline{D}$, then the solutions to (1) are pathwise unique (Saisho [4]). In this talk, we prove the pathwise uniqueness under certain conditions which allow non-Lipschitz conefficients σ and b. Under the conditions, the solutions to (1) can explode. So, we also provide two types of conditions for the solutions to be non-explosive.

In what follows, we give a precise setting and explain the main results. We denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ the standard inner product and norm on $\mathbb{R}^d$, respectively. We write $\|\cdot\|$ for the Hilbert–Schmidt norm on $\mathbb{R}^d\otimes\mathbb{R}^d$. For $x\in\partial D(=\overline{D}\setminus D)$ and $r\in(0,\infty)$, we define

$$\mathcal{N}_{x,r} = \{\mathbf{n}\in\mathbb{R}^d \mid |\mathbf{n}|=1,\ B(x-r\mathbf{n},r)\cap D=\emptyset\},\quad \mathcal{N}_x = \bigcup_{r\in(0,\infty)}\mathcal{N}_{x,r}.$$

Here, $B(x,r)$ denotes the open ball in $\mathbb{R}^d$ centered at x with radius r. Following [3, 4], we introduce conditions **(A)** and **(B)** on D as follows:

(A) There exists $r_0\in(0,\infty)$ such that $\mathcal{N}_x = \mathcal{N}_{x,r_0}\neq\emptyset$ for any $x\in\partial D$.

(B) There exist $\delta\in(0,\infty)$ and $\beta\in[1,\infty)$ with the requirement that for any $x\in\partial D$ there exists a unit vector $\mathbf{1}_x$ such that $\langle\mathbf{1}_x,\mathbf{n}\rangle\geq 1/\beta$ for any $\mathbf{n}\in\bigcup_{y\in B(x,\delta)\cap\partial D}\mathcal{N}_y$.

In order to give a sufficient condition for the pathwise uniqueness, we introduce the following condition for a nonnegative and Borel measurable function Λ on $[0,1)$:

(L) There exists $\varepsilon_0\in(0,1)$ such that Λ is continuous and non-decreasing on $[0,\varepsilon_0)$, and $\int_0^{\varepsilon_0}\Lambda(s)^{-1}\,ds=\infty$.

For example, $\Lambda(s)=s$ and $\Lambda(s)=s\log(1/s)$ satisfy these conditions. Let $g=\{g(t,\cdot)\}_{t\geq 0}$ be a nonnegative progressively measurable process such that $P(\int_0^T g(s,\cdot)\,ds<\infty)=1$ for any $T>0$. Then, a sufficient condition for the pathwise uniqueness is described as follows.

Theorem 1. Assume condition **(A)** and that for each $R>0$ there exists a Borel measurable function $\Lambda_R : [0,1)\to\mathbb{R}_+$ satisfying **(L)** such that for P-a.s. ω,

$$\|\sigma(t,\omega,x)-\sigma(t,\omega,y)\|^2 + 2\langle x-y, b(t,\omega,x)-b(t,\omega,y)\rangle \leq g(t,\omega)\Lambda_R(|x-y|^2)$$

for any $t\geq 0$ and $x,y\in\overline{D}\cap B(0,R)$ with $|x-y|<1$. Then, we have $P(X(t)=Y(t),\ t\geq 0)=1$ for any solutions $X=\{X(t)\}_{t\geq 0}$ and $Y=\{Y(t)\}_{t\geq 0}$ to (1) with the same Brownian motion $B=\{B(t)\}_{t\geq 0}$.

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Theorem 1 can be regarded as a natural generalization of [2, Theorem 2] and inherit the assumptions in that work. Note that coefficients σ and b satisfying the condition in Theorem 1 can be non-Lipschitz continuous.

Next, we discuss the non-explosion property of the solution. Let $\gamma\colon \mathbb{R}_+ \to [1,\infty)$ be a continuous and non-decreasing function such that $\lim_{s\to\infty}\gamma(s)=\infty$ and $\int_0^\infty \gamma(s)^{-1}\,ds=\infty$. We note that functions $\gamma(s)=s+1$, $\gamma(s)=s\log(s+1)+1$ are typical examples satisfying these conditions. For a solution X to (1), we set $\zeta_X=\inf\{t>0 \mid X(t)\notin \overline{D}\}$.

Theorem 2. Assume condition **(A)**, and that there exists a nonnegative function $V\in C^{1,2}([0,\infty)\times\mathbb{R}^d)$ with the following conditions:

(2.1) For any $t>0$,

$$\lim_{R\to\infty}\inf_{s\in[0,t],\,x\in\overline{D}\setminus B(0,R)} V(s,x)=\infty.$$

(2.2) For any $x\in\partial D$, $t\ge 0$, and $\mathbf{n}\in\mathcal{N}_x$, $\langle(\nabla V)(t,x),\mathbf{n}\rangle\le 0$.

(2.3) For P-a.s. ω,

$$\begin{aligned}&\|\sigma(t,\omega,x)\|^2(\Delta V)(t,x)+2\langle b(t,\omega,x),(\nabla V)(t,x)\rangle+2\frac{\partial V}{\partial t}(t,x)\\&\le g(t,\omega)\gamma(V(t,x))\end{aligned}$$

for any $t\ge 0$ and $x\in\overline{D}$.

Then, the solutions to (1) are non-explosive. That is, $P(\zeta_X=\infty)=1$ for any solution X to (1).

Theorem 2 corresponds to [2, Theorem 1], where the non-explosion property of SDEs without reflection terms was discussed. In the talk, we will explain that these conditions fit for convex domains and some domains whose boundaries are (globally) described by smooth functions. For such domains, we will also confirm that the norms of coefficients σ and b in (1) can grow as $|x|(\log|x|)^{1/2}$ in x as $|x|\to\infty$.

Theorem 3 below describes the non-explosion property for more general domains but with more restrictive conditions on the coefficients. For $x\in\overline{D}$, $\delta>0$, and $T>0$, we define

$$M(x,\delta,T)=\operatorname*{ess\,sup}_{\omega\in\Omega}\sup_{t\in[0,T],\,z\in B(x,\delta)\cap\overline{D}}\{\|\sigma(t,\omega,z)\|^2\vee|b(t,\omega,z)|^2\}.$$

Then, we have the following.

Theorem 3. Assume conditions **(A)** and **(B)**, and the following:

(3.1) For P-a.s. ω,

$$\|\sigma(t,\omega,x)\|^2\vee|b(t,\omega,x)|^2\le g(t,\omega)\gamma(|x|^2)$$

for any $t\ge 0$ and $x\in\overline{D}$.

(3.2) For each $T>0$, there exist constants $C>0$, $\nu\in[0,1)$, $\hat{\delta}>0$, $\hat{\beta}\in(0,1)$, points $\{x_n\}_{n=1}^\infty\subset\partial D$, and positive numbers $\{\delta_n\}_{n=1}^\infty\subset[\hat{\delta},\infty)$ such that $\partial D\subset\bigcup_{n=1}^\infty B(x_n,\hat{\beta}\delta_n)$ and

$$M(x_n,\delta_n,T)\le C\delta_n^\nu\quad\text{for any } n\in\mathbb{N}.$$

Then, the solutions to (1) are non-explosive.

In the talk, we will explain that Theorem 3 allows the coefficients to have at most sub-linear growth of order $(1/2)-\varepsilon$ in x for some $\varepsilon>0$ as $|x|\to\infty$.

References

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