Stochastic quantization associated with the $\exp(\Phi)_2$-quantum field model driven by space-time white noise on the torus in the full L^1-regime

Seiichiro Kusuoka
(Department of Mathematics, Kyoto University)

This talk is based on [2], which is a joint work with Masato Hoshino (Kyushu University) and Hiroshi Kawabi (Keio University).

Let Λ be the 2-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ and μ_0 be Nelson's free field measure on Λ with mass 1. We consider the $\exp(\Phi)_2$-quantum field measure:

$$\mu^{(\alpha)}(d\phi) = \frac{1}{Z^{(\alpha)}} \exp\left(- \int_\Lambda \exp^\diamond(\alpha\phi)(x)dx \right) \mu_0(d\phi)$$

where $Z^{(\alpha)}$ is the normalizing constant and $\exp^\diamond(\alpha\phi)$ is the renormalized term of $\exp(\alpha\phi)$ under $\mu_0(d\phi)$. The stochastic quantization of $\mu^{(\alpha)}$ is given by the SPDE

$$\partial_t \Phi_t(x) = \frac{1}{2}(\triangle - 1)\Phi_t(x) - \frac{\alpha}{2}\exp^\diamond(\alpha\Phi_t(x)) + \dot{W}_t(x), \quad x \in \Lambda \tag{1}$$

where $\dot{W}_t(x)$ is a white noise with parameter (t, x). We denote the Besov space and the Sobolev space on Λ by $B^s_{p,r}(\Lambda)$ and $W^{s,p}(\Lambda)$, respectively. Let $H^s(\Lambda) := W^{s,2}(\Lambda)$. In this talk we consider (1) for $\alpha \in (-\sqrt{8\pi}, \sqrt{8\pi})$, while in the previous work [1] we assumed that $\alpha \in (-\sqrt{4\pi}, \sqrt{4\pi})$. It is known that for $\alpha \in (-\sqrt{4\pi}, \sqrt{4\pi})$, we are able to construct the Gaussian multiplicative chaos $\exp^\diamond(\alpha\phi)$, which is a distribution for μ_0-almost every ϕ, in $L^2(\mu_0; \mathcal{D}'(\Lambda))$, however for $\alpha \in (-\sqrt{8\pi}, \sqrt{8\pi}) \setminus (-\sqrt{4\pi}, \sqrt{4\pi})$ we have to treat it in $L^p(\mu_0; \mathcal{D}'(\Lambda))$ for some $p \in (1, 2)$. This difference makes the problem much more difficult. For example, the Wiener chaos expansion does not work very well in $L^p(\mu_0; \mathcal{D}'(\Lambda))$ for $p \in (1, 2)$. The same problem also appears in solving the stochastic quantization equation (1).

For the renormalization, we introduce an approximation operator P_N. Let $\psi : \mathbb{R}^2 \to [0, 1]$ be a Borel measurable function satisfying the following properties.

(i) $\psi(0) = 1$ and $\psi(x) = \psi(-x)$ for any $x \in \mathbb{R}^2$.

(ii) $\sup_{x\in\mathbb{R}^2} |x|^{2+\kappa}|\psi(x)| < \infty$ for some $\kappa > 0$.

(iii) $\sup_{x\in\mathbb{R}^2} |x|^{-\zeta}|\psi(x) - 1| < \infty$ for some $\zeta > 0$.

Note that ψ need not to be continuous except the origin. For a such function ψ, we define the approximation operator P_N on $\mathcal{D}'(\Lambda)$ by

$$P_N f = \sum_{k\in\mathbb{Z}^2} \psi(2^{-N}k)\hat{f}(k)e_k, \qquad f \in \mathcal{D}'(\Lambda),\ N \in \mathbb{N},$$

where $\{e_k; k \in \mathbb{Z}^2\}$ is the Fourier basis and $\hat{f}(k)$ is the Fourier coefficient of f with $k \in \mathbb{Z}^2$. Let

$$C_N := \frac{1}{4\pi^2} \sum_{k\in\mathbb{Z}^2} \frac{\psi(2^{-N}k)^2}{1 + |k|^2}.$$

For such a general approximation operator P_N, we proved in [2] the convergence of $\exp(\alpha P_N\phi - (\alpha^2/2)C_N)$ in a suitable Besov space with a negative index for μ_0-almost every ϕ. We remark that the limit is called the Gaussian multiplicative chaos. By using this convergence we obtain results on the solutions to the stochastic quantization equation (1), which are similar to those in [1].

Theorem 1. *Let $|\alpha| < \sqrt{8\pi}$, $p \in (1, (8\pi/\alpha^2) \wedge 2)$, and $\varepsilon > 0$. For any $N \in \mathbb{N}$, consider the initial value problem*

$$\begin{cases} \partial_t \Phi_t^N = \frac{1}{2}(\triangle - 1)\Phi_t^N - \frac{\alpha}{2}\exp\left(\alpha\Phi_t^N - \frac{\alpha^2}{2}C_N\right) + P_N \dot{W}_t, & t > 0, \\ \Phi_0^N = P_N\phi, \end{cases} \tag{2}$$

where $\phi \in \mathcal{D}'(\Lambda)$. Then for μ_0-almost every $\phi \in \mathcal{D}'(\Lambda)$, the unique time-global solution Φ^N converges to a $B_{p,p}^{-\varepsilon}(\Lambda)$-valued stochastic process Φ in the space $C([0,T]; B_{p,p}^{-\varepsilon}(\Lambda))$ for any $T > 0$, $\mathbb{P}$-almost surely. Moreover, the limit Φ is independent of the choice of ψ.

Now we add some assumptions on P_N as follows.

(i) P_N is nonnegative, that is, $P_N f \geq 0$ for $f \geq 0$.

(ii) For any $p \in (1, 2)$, $s \in \mathbb{R}$, there exists a constant $C > 0$ such that

$$\sup_{N\in\mathbb{N}} \|P_N f\|_{B_{p,p}^s} \leq C\|f\|_{B_{p,p}^s}, \qquad \lim_{N\to\infty} \|P_N f - f\|_{B_{p,p}^s} = 0$$

for any $f \in B_{p,p}^s(\Lambda)$.

We remark that mollifiers with nonnegative smooth functions satisfy the assumptions on P_N. Under the extra assumptions on P_N, we have the following theorem.

Theorem 2. *Let $|\alpha| < \sqrt{8\pi}$ and $\varepsilon > 0$. For any $N \in \mathbb{N}$, consider the solution $\mathbf{\Phi}^N = \mathbf{\Phi}^N(\phi)$ of the SPDE*

$$\begin{cases} \partial_t \mathbf{\Phi}_t^N = \frac{1}{2}(\triangle - 1)\mathbf{\Phi}_t^N - \frac{\alpha}{2}P_N\exp\left(\alpha P_N\mathbf{\Phi}_t^N - \frac{\alpha^2}{2}C_N\right) + \dot{W}_t, & t > 0, \\ \mathbf{\Phi}_0^N = \phi \in \mathcal{D}'(\Lambda). \end{cases} \tag{3}$$

Let ξ_N be a random variable with the law $\mu_N^{(\alpha)}$ independent of W. Then $\bar{\mathbf{\Phi}}^N = \mathbf{\Phi}(\xi_N)$ is a stationary process and converges in law as $N \to \infty$ to Φ obtained in Theorem 1 with an initial law $\mu^{(\alpha)}$, on the space $C([0,T]; H^{-\varepsilon}(\Lambda))$ for any $T > 0$. Moreover, the law of the random variable $\bar{\mathbf{\Phi}}_t$ is $\mu^{(\alpha)}$ for any $t \geq 0$.

Finally, we state a result on the Dirichlet form with respect to $\mu^{(\alpha)}$.

Theorem 3. *Let $|\alpha| < \sqrt{8\pi}$. The pre-Dirichlet form $(\mathcal{E}, \mathfrak{F}C_b^\infty)$ given by*

$$\mathcal{E}(F,G) = \frac{1}{2}\int_E \langle DF(\phi), DG(\phi)\rangle \mu^{(\alpha)}(d\phi), \quad F, G\text{:cylindrical smooth functions,}$$

where D is the Gateau derivative operator on $L^2(\Lambda)$, is closable in $L^2(H^{-s}(\Lambda), \mu^{(\alpha)})$ for some $s \in (0,1)$. Denote the Markov process associated to the closure of $(\mathcal{E}, \mathfrak{F}C_b^\infty)$ by $(\Psi, (Q_\phi)_{\phi\in H^{-s}})$. Then for $\mu^{(\alpha)}$-almost every ϕ, the diffusion process Ψ coincides $\mathbb{Q}_\phi$-almost surely with Φ obtained in Theorem 1 with the initial value ϕ, driven by some $L^2(\Lambda)$-cylindrical $(\mathcal{G}_t)_{t\geq 0}$-Brownian motion $\mathcal{W} = (\mathcal{W}_t)_{t\geq 0}$.

[1] M. Hoshino, H. Kawabi and S. Kusuoka, Stochastic quantization associated with the $\exp(\Phi)_2$-quantum field model driven by space-time white noise on the torus, to appear in *Journal of Evolution Equations*, arXiv: 1907.07921.

[2] M. Hoshino, H. Kawabi and S. Kusuoka, Stochastic quantization associated with the $\exp(\Phi)_2$-quantum field model driven by space-time white noise on the torus in the full L^1-regime, arXiv:2007.08171.