UPPER DENSITY ESTIMATES FOR THE MARGINAL LAW OF AN STABLE PROCESS AND ITS MAXIMUM: FROM SIMULATION TO THEORY

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1. Abstract

Infinite dimensional analysis is already a well established subject. In most cases, like in the Wiener case, one uses the independent increment property of the underlying process to establish a differential calculus by closing finite dimensional derivative operators.

A similar structure is used for jump type processes with the exception that there are two possible choices: using the jump size or the jump times of the underlying process.

At any rate, the methodologies that have been available so far for jump process are not well adapted in order to obtain optimal estimates. On the other hand, they are good in order to consider very general functionals.

In this presentation, we consider a non-symmetric α-stable process X and its supremum in a closed time interval $S_T := \sup_{[0,T]} X_s$. In this presentation, we will explain our results on

- The existence, uniqueness and smoothness of the joint density of (X_T, S_T) in its domain
- Upper bounds for the density and mixed derivatives

The study of the law of the supremum of stable processes has a long history starting from such as Darling (1956), Heyde (1969) and Bingham (1973). The progress in this problem has been progressive. Most of the detailed results on the law of the supremum of a Lévy process do not rely on the Wiener-Hopf factorization but rather on equivalences with laws related to excursions of reflected processes.

For example, in [2], the author obtains explicit formulas for the supremum and the time this supremum is achieved in the case of spectrally negative stable process. Also some other related formulas are obtained in the symmetric Cauchy case. The continuity of the density of the supremum was obtained in [3].

Expansions of these densities have been achieved in [6] and [7]. These results are usually obtained by linking the Wiener-Hopf factorisation with elliptic type functions. For this reason, this methodology only works for certain particular type of parameter combinations. Furthermore extracting information from these series expansions is always a challenge as the coefficients are expressed through complex expressions.

Obviously these problems are also related to first passage probabilities but we will not discuss this further here.

The results that interested us in this problem appear in [5] and [4]. In these articles the authors discuss the asymptotic behavior of the law of the sumpremum at infinity and at zero. The authors use either the Wiener-Hopf factorisation result or excursion theory in order to obtain their results.

The result where our research started is [5] which relies on excursion type results for the supremum process. For this reason, it is difficult to foresee that such an approach will be successful to analyze the law of (X_T, S_T).

Normally, the density of a stable process has two difference space-time regimes, one near zero and another at infinity. Similarly, by looking at the results in [5], one observes two different regimes. Then one expects that the density will have four different regimes and is strictly positive only in the upper diagonal set determined by $y = x$, $y \geq 0$.

Our main results gives the characterization of the density in the above set and it gives details of the density such as the cases when the density explodes at the boundary $y = x$ and the cases it is bounded at the boundary and the rate of explosion.

Our result falls slightly short of being optimal due to the use of a Chebyshev's type argument to obtain the upper bound.

In this article, we obtain integration by parts formulas for the joint law of a stable process and its maximum. The argument is based on a multi-level representation for the joint law which uses the theory of convex majorants for stable processes and the Chambers-Mallows-Stuck representation for stable random variables.

The only parallel result using infinite dimensional analysis was obtained in [1], where the authors obtained the existence of the density for the supremum of a random variable based on a stochastic equation driven by a jump process.

References

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