COMMUTATOR ESTIMATES FROM A VIEWPOINT OF REGULARITY STRUCTURES

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1. Introduction

Both the *regularity structures* (RS) [4] and the *paracontrolled calculus* (PC) [3] are general theories that give rigorous meanings to singular stochastic PDEs. Both have their roots in the rough path theory for ODEs driven by irregular controls. The RS provides a "microscopic" pointwise description of dynamics, while the PC a "macroscopic" spectral description. In fact, these two descriptions are equivalent, as clarified in the papers [1, 2].

In this talk, we introduce an application of [1, 2] to the *commutator estimates*, which have important roles in the PC. Recall from [3] the definition of the bilinear operator $\prec$ called *Bony's paraproduct*. We define the multilinear operators as follows.

Definition 1. *For any functions $\xi, f_1, f_2, \dots$ in $\mathcal{S}(\mathbb{R}^d)$, define*

$$\begin{aligned} C_1(f_1,\xi) &:= f_1 \succeq \xi \ (:= f_1\xi - f_1 \prec \xi), \\ C_n(f_1,\dots,f_n,\xi) &:= C_{n-1}(f_1 \prec f_2, f_3,\dots,f_n,\xi) - f_1 C_{n-1}(f_2,f_3,\dots,f_n,\xi). \end{aligned}$$

For each $\alpha \in \mathbb{R}$, let C_0^α be the completion of $\mathcal{S}(\mathbb{R}^d)$ in the Besov norm $C^\alpha = B^\alpha_{\infty,\infty}$. The following theorem is a multivariable extension of the original commutator estimate ([3, Lemma 2.4]).

Theorem 1 ([5, Theorem 4.1])**.** *Let $\alpha_1,\dots,\alpha_n \in (0,1)$ and $\alpha_\circ < 0$ be the regularity exponents such that*

$$\alpha_1 + \cdots + \alpha_n < 1, \qquad \alpha_2 + \cdots + \alpha_n + \alpha_\circ < 0 < \alpha_1 + \cdots + \alpha_n + \alpha_\circ.$$

Then the operator C_n is uniquely extended to the continuous operator from $C_0^{\alpha_1} \times \cdots \times C_0^{\alpha_n} \times C_0^{\alpha_\circ}$ to $C_0^{\alpha_1+\cdots+\alpha_n+\alpha_\circ}$.

Theorem 1 was extended to the $B^\alpha_{p,q}$-type norms with arbitrary $p, q \in [1,\infty]$ in [6].

2. Proof of Theorem 1

In what follows, we consider the "word Hopf algbera", which is also important in the geometric rough path theory. Fix $n \in \mathbb{N}$ and define the set of "words"

$$W := \{(k, k+1, \dots, \ell-1, \ell) \ ; \ 1 \le k \le \ell \le n\}.$$

Denote by $\mathbb{R}[W]$ the polynomial ring in W, and $\mathbf{1}$ its unit.

Proposition 2. *Define the algebra morphism $\Delta : \mathbb{R}[W] \to \mathbb{R}[W] \otimes \mathbb{R}[W]$ by $\Delta\mathbf{1} = \mathbf{1} \otimes \mathbf{1}$ and*

$$\Delta(k,\dots,\ell) = (k,\dots,\ell) \otimes \mathbf{1} + \mathbf{1} \otimes (k,\dots,\ell) + \sum_{m=k}^{\ell-1} (k,\dots,m) \otimes (m+1,\dots,\ell)$$

for any $1 \le k \le \ell \le n$. Then $\mathbb{R}[W]$ is a Hopf algebra.

We define the "word regularity structure" as follows. On the basis of the linear space $\mathbb{R}[W]$ (i.e., the monomials of words), we define the grading $|\cdot|$ by

$$|\tau| := \begin{cases} 0, & \tau = \mathbf{1}, \\ \alpha_k + \cdots + \alpha_\ell, & \tau = (k, \dots, \ell), \end{cases}$$

and by extending it multiplicatively. Then $T^+ = \mathbb{R}[W]$ is a graded Hopf algebra. Moreover, on the linear subspace $T = \langle W \cup \{\mathbf{1}\} \rangle$ we can define another grading $|\tau|' := |\tau| + \alpha_0$ for each $\tau \in W \cup \{\mathbf{1}\}$. Then T is a graded linear space with the comodule structure $\Delta : T \to T^+ \otimes T$, so the pair (T^+, T) is a *concrete regularity structure* ([1, 2]). Recall the definition of *models* from these references.

Theorem 3 ([2, Theorem 1])**.** *Let $\mathscr{M}$ be the set of all models on the concrete regularity structure (T^+, T). For any $(\Pi, g) \in \mathscr{M}$ and $\sigma, \tau \in W \cup \{\mathbf{1}\}$, we can define the functions/distributions $[g]\sigma \in C^{|\sigma|}$ and $[\Pi]\tau \in C^{|\tau|'}$ (continuously depending on (Π, g)) by*

$$g(\sigma) = \sum_{\sigma_1, \sigma_2 \neq \mathbf{1}} g(\sigma_1) \prec [g]\sigma_2 + [g]\sigma, \qquad \Pi\tau = \sum_{\tau_1 \neq \mathbf{1}} g(\tau_1) \prec [\Pi]\tau_2 + [\Pi]\tau,$$

where (σ_1, σ_2) runs over all partition of σ, that is, $\sigma_1, \sigma_2 \in W \cup \{\mathbf{1}\}$ and $\sigma_1 \otimes \sigma_2$ appears in the minimal expansion of $\Delta\sigma$. Moreover, the mapping

$$\mathscr{M} \ni (\Pi, g) \mapsto \left(([\Pi]\tau)_{\substack{\tau \in W \cup \{\mathbf{1}\} \\ |\tau|' < 0}}, ([g]\sigma)_{\sigma \in W} \right) \in \prod_{\substack{\tau \in W \cup \{\mathbf{1}\} \\ |\tau|' < 0}} C^{|\tau|'} \times \prod_{\sigma \in W} C^{|\sigma|}.$$

is a bijection and locally bi-Lipschitz continuous.

By Theorem 3, for given data $(f_1, \dots, f_n, \xi) \in C_0^{\alpha_1} \times \cdots \times C_0^{\alpha_n} \times C_0^{\alpha_\circ}$, we can define the model (Π, g) by

$$[g]\sigma = \begin{cases} f_k, & \sigma = (k), \\ 0, & \sigma = (k, \dots, \ell),\ k < \ell. \end{cases}, \qquad [\Pi]\tau = \begin{cases} \xi, & \tau = \mathbf{1}, \\ 0, & \tau \in W,\ |\tau|' < 0. \end{cases}$$

Then we can complete the proof of Theorem 1, since the composition of mappings

$$(f_1, \dots, f_n, \xi) \mapsto ([\Pi], [g]) \mapsto (\Pi, g) \mapsto [\Pi](1, \dots, n)$$

coincides with the commutator $C_n(f_1, \dots, f_n, \xi)$, if all functions $f_1, \dots, f_n, \xi$ are smooth.

References

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