# Parametrix method for multi-skewed Brownian motion

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Let $x_0 \in \mathbf{R}$, $n \in \mathbf{N}$, $\alpha_1, \ldots, \alpha_n \in (-1, 1)$ and $-\infty < a_1 < a_2 < \cdots < a_n < +\infty$. We consider one dimensional SDEs of the form

$$X_t(x_0) = x_0 + B_t + \sum_{i=1}^{n} \alpha_i L_t^{a_i}(X), \tag{1}$$

where $\{B_t\}_{t \geq 0}$ is a one-dimensional Brownian motion and $L_t^{a_i}(X)$ denotes the symmetric local time of $X$ at the point $a_i$ until the time $t$. If $n = 1$ and $a_1 = 0$, the process $X$ is called the skew Brownian motion. In [3], one can find exact simulation methods for the skew Brownian motion. These methods have been extended to some other cases in [1] using resolvent methods. In the case of $n = 2$, a simulation method has been proposed in [2] which points out at the difficulty of obtaining exact simulations methods for $n \geq 3$. In this talk, we propose a simulation method for any $n$. The method is based on an expansion for $\mathbf{E}[f(X_t(x_0))]$ which is obtained by the parametric method.

## Main Result

Let us define $b_i := \frac{a_i + a_{i+1}}{2}$ for $i = 1, \cdots, n-1$. Fix $0 < \varepsilon < \min_{1 \leq i \leq n-1} \{\frac{a_{i+1} - a_i}{2}\}$ arbitrary. Let $\varphi_1, \ldots, \varphi_n$ be elements of $C_b^2(\mathbf{R})$ which satisfy the following conditions.

(H1): $\sum_{i=1}^{n} \varphi_i \equiv 1$.

(H2): $\operatorname{supp} \varphi_1 = (-\infty, b_1 + \varepsilon]$, $\varphi_1 = 1$ on $(-\infty, b_1 - \varepsilon]$ and decreasing on $(b_1 - \varepsilon, b_1 + \varepsilon]$.

(H3): For $2 \leq i \leq n-1$, $\operatorname{supp} \varphi_i = [b_{i-1} - \varepsilon, b_i + \varepsilon]$, increasing on $(b_{i-1} - \varepsilon, b_{i-1} + \varepsilon]$, $\varphi_i = 1$ on $[b_{i-1} + \varepsilon, b_i - \varepsilon]$ and decreasing on $(b_i - \varepsilon, b_i + \varepsilon]$.

(H4): $\operatorname{supp}\varphi_n = [b_{n-1}-\varepsilon, +\infty)$, $\varphi_n = 1$ on $[b_{n-1}+\varepsilon, +\infty)$ and increasing on $[b_{n-1}-\varepsilon, b_{n-1}+\varepsilon)$.

Let $\widetilde{X}^i(x_0)$ be the solution to the SDE

$$\widetilde{X}^i_t(x_0) = x_0 + B_t + \alpha_i L^{a_i}_t(\widetilde{X}^i(x_0)). \tag{2}$$

For a bounded Borel measurable function $f$, we put

$$P_t f(x) := \mathbf{E}[f(X_t(x))] \text{ and } \widetilde{P}_t f(x) := \sum_{i=1}^n \mathbf{E}[\varphi_i(\widetilde{X}^i_t(x)) f(\widetilde{X}^i_t(x))] \ (t \in (0,T]).$$

**Theorem 0.1.** *Let $x_0 \in \mathbf{R}$, $X$ be a solution to (1) and for $i = 1, \cdots, n$, $\widetilde{X}^i$ be a solution to (2). Then for $t \in (0,T]$, we have that*

(3)
$$P_t f(x_0) = \widetilde{P}_t f(x_0) + \sum_{i=1}^n \int_0^t \mathbf{E}\left[\widetilde{P}_{t-s} f(\widetilde{X}^i_s(x_0)) \Theta^i_s(x_0, \widetilde{X}^i_s(x_0))\right] ds$$
$$+ \sum_{m=2}^\infty \sum_{1 \le i_1, \cdots, i_m \le n} \int_{\Delta_m(t)} \mathbf{E}\left[\widetilde{P}_{t-\sum_{l=1}^m s_l} f(\widetilde{Y}^{i_m}_{s_m}) \prod_{j=1}^m \Theta^{i_j}_{s_j}(\widetilde{Y}^{i_{j-1}}_{s_{j-1}}, \widetilde{Y}^{i_j}_{s_j})\right] \mathbf{ds^m_1},$$

*where $\widetilde{Y}^{i_0}_{s_0} := x_0$, $\widetilde{Y}^{i_j}_{s_j} := \widetilde{X}^{i_j}_{s_j}(\widetilde{Y}^{i_{j-1}}_{s_{j-1}})$ for $j \ge 1$, $\Theta^i_s(x_0, x) := \frac{1}{2}\left(\varphi''_i(x) + 2\varphi'_i(x)\frac{\partial_x p^i_s(x_0,x)}{p^i_s(x_0,x)}\right)$ and $p^i$ denotes the transition density function of $\widetilde{X}^i$.*

# References


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[3] Pierre Etoré and Miguel Martinez. "Exact simulation of one-dimensional stochastic differential equations involving the local time at zero of the unknown process." Monte Carlo Methods and Applications, 19(1):41–71, 2013.