Applications of non-local Dirichlet forms defined on infinite dimensional spaces

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Let the state space S be a weighted l^p space, denoted by $l^p_{(\beta_i)}$, such that, for some $p \in [1, \infty)$ and a weight $(\beta_i)_{i \in \mathbb{N}}$ with $\beta_i \geq 0, i \in \mathbb{N}$,

$$S = l^p_{(\beta_i)} \equiv \Big\{ \mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} : \|\mathbf{x}\|_{l^p_{(\beta_i)}} \equiv \Big(\sum_{i=1}^{\infty} \beta_i |x_i|^p \Big)^{\frac{1}{p}} < \infty \Big\}, \qquad (1)$$

or a weighted l^∞ space, denoted by $l^\infty_{(\beta_i)}$, such that for a weight $(\beta_i)_{i \in \mathbb{N}}$ with $\beta_i \geq 0, i \in \mathbb{N}$,

$$S = l^\infty_{(\beta_i)} \equiv \Big\{ \mathbf{x} = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} : \|\mathbf{x}\|_{l^\infty_{(\beta_i)}} \equiv \sup_{i \in \mathbb{N}} \beta_i |x_i| < \infty \Big\}, \qquad (2)$$

or

$$S = \mathbb{R}^{\mathbb{N}}, \tag{3}$$

the direct product space with the metric $d(\cdot,\cdot)$ such that for $\mathrm{x},\ \mathrm{x}' \in \mathbb{R}^{\mathbb{N}}$, $d(\mathrm{x},\mathrm{x}') \equiv \sum_{k=1}^{\infty} (\frac{1}{2})^k \frac{\|\mathrm{x}-\mathrm{x}'\|_k}{\|\mathrm{x}-\mathrm{x}'\|_k+1}$, with $\|\mathrm{x}\|_k = (\sum_{i=1}^{k} (x_i)^2)^{\frac{1}{2}},\ \mathrm{x} = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}}$.
Denote by $\mathcal{B}(S)$ the Borel σ-field of S. Let μ be a given Borel probability measure on $(S, \mathcal{B}(S))$. For each $i \in \mathbb{N}$, let σ_{i^c} be the sub σ-field of $\mathcal{B}(S)$ that is generated by the Borel sets

$$B = \left\{ \mathrm{x} \in S \ \middle|\ x_{j_1} \in B_1, \dots x_{j_n} \in B_n \right\}, \tag{4}$$

$j_k \neq i,\ B_k \in \mathcal{B}^1,\ k = 1, \dots, n,\ n \in \mathbb{N},$ where $\mathcal{B}^1$ denotes the Borel σ-field of $\mathbb{R}^1$, For each $i \in \mathbb{N}$, let $\mu(\cdot \,|\, \sigma_{i^c})$ be the conditional probability, a one-dimensional probability distribution-valued σ_{i^c} measurable function, that is characterized by

$$\mu\big(\{\mathrm{x} \ : \ x_i \in A\} \cap B\big) = \int_B \mu(A \,|\, \sigma_{i^c})\, \mu(d\mathrm{x}), \qquad (5)$$

$\forall A \in \mathcal{B}^1, \ \forall B \in \sigma_{i^c}$. Define

$$L^2(S;\mu) \equiv \Big\{ f \ \Big| \ f : S \to \mathbb{R}, \text{ measurable and}$$

$$\|f\|_{L^2} = \Big(\int_S |f(\mathrm{x})\mu(d\mathrm{x})\Big)^{\frac{1}{2}} < \infty \Big\}, \qquad (6)$$

also define

$$\mathcal{F}C_0^\infty \equiv \text{the } \mu \text{ equivalence class of}$$

$$\Big\{ f \ \Big| \ \exists n \in \mathbb{N}, \ f \in C_0^\infty(\mathbb{R}^n \to \mathbb{R}) \Big\} \subset L^2(S;\mu). \qquad (7)$$

On $L^2(S;\mu)$, for $0 < \alpha < 2$, define the Markovian symmetric forms $\mathcal{E}_{(\alpha)}$ called *individually adapted Markovian symmetric form of index α to the measure μ*:

Firstly, for each $0 < \alpha < 2$ and $i \in \mathbb{N}$, and for the variables $y_i,\ y_i' \in \mathbb{R}^1$, $\mathrm{x} = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) \in S$ and $\mathrm{x} \setminus x_i \equiv (x_1, \dots, x_{i-1}, x_{i+1}, \dots)$, let

$$\begin{aligned}
&\Phi_\alpha(u, v; y_i, y_i', \mathrm{x} \setminus x_i) \\
&\qquad \equiv \frac{1}{|y_i - y_i'|^{2\alpha+1}} \times \Big\{ u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots) \\
&\qquad - u(x_1, \dots, x_{i-1}, y_i', x_{i+1}, \dots) \Big\} \\
&\qquad \times \Big\{ v(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots) \\
&\qquad - v(x_1, \dots, x_{i-1}, y_i', x_{i+1}, \dots) \Big\},
\end{aligned} \tag{8}$$

then, for each $0 < \alpha \leq 1$ and $i \in \mathbb{N}$, define

$$\mathcal{E}^{(i)}_{(\alpha)}(u,v) \equiv \int_S \Big\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}}(y_i)\, \Phi_\alpha(u,v;y_i,x_i,\mathrm{x} \setminus x_i) \times \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d\mathrm{x}). \tag{9}$$

and

$$\mathcal{E}_{(\alpha)}(u,v) \equiv \sum_{i \in \mathbb{N}} \mathcal{E}^{(i)}_{(\alpha)}(u,v), \tag{10}$$

where $I_{\{\cdot\}}$ denotes the indicator function.

For $y_i \neq y_i'$, (8) is well defined for any real valued $\mathcal{B}(S)$-measurable functions u and v. For *the Lipschiz continuous* **functions $\tilde{u} \in C_0^{\infty}(\mathbb{R}^n \to \mathbb{R}) \subset \mathcal{F}C_0^{\infty}$ resp. $\tilde{v} \in C_0^{\infty}(\mathbb{R}^m \to \mathbb{R}) \subset \mathcal{F}C_0^{\infty}$, $\ n, m \in \mathbb{N}$ which are representations of $u \in \mathcal{F}C_0^{\infty}$ resp. $v \in \mathcal{F}C_0^{\infty}$, $\ n, m \in \mathbb{N}$, (9) and (10) are well defined (the right hand side of (10) has only a finite number of sums). In Theorem 1 given below we see that (9) and (10) are well defined for $\mathcal{F}C_0^{\infty}$, the space of μ-equivalent class.**

For $1 < \alpha < 2$, we suppose that for each $i \in \mathbb{N}$, the conditional distribution $\mu(\cdot \,|\, \sigma_{i^c})$ can be expressed by a locally bounded probability density $\rho(\cdot \,|\, \sigma_{i^c})$, μ-a.e.. Precisely (cf. (2.5) of [AR91]), there exists a σ_{i^c}-measurable function $0 \leq \rho(\cdot \,|\, \sigma_{i^c})$ on $\mathbb{R}^1$ and

$$\mu(dy \,|\, \sigma_{i^c}) = \rho(y \,|\, \sigma_{i^c})\, dy, \quad \mu - a.e., \tag{11}$$

holds, with $\rho(\cdot \,|\, \sigma_{i^c})$ a function such that for any compact $K \subset \mathbb{R}$ there exists an $L_i < \infty$, which may depend on i, and for any $y \in K$,

$$\operatorname{ess\,sup}_{y \in \mathbb{R}^1} \rho(y \,|\, \sigma_{i^c}) \leq L_i, \qquad \mu - a.e., \tag{12}$$

where $\operatorname{ess\,sup}_{y \in \mathbb{R}^1}$ is taken with respect to the Lebesgue measure on $\mathbb{R}^1$. Then define the non-local form $\mathcal{E}_{(\alpha)}(u, v)$, for $1 < \alpha < 2$, by the same formula as (10).

Remark 1. *For the $\mathcal{B}(S)$ measurable function $\int_{\mathbb{R}} \mathbf{1}_{\{y_i \neq x_i\}} \Phi_\alpha(u, v; y_i, x_i, \mathrm{x} \setminus x_i) \mu(dy_i \mid \sigma_{i^c})$ by taking the expectation conditioned by the sub σ-field σ_{i^c}, it holds that (cf.,[Fukushima, Uemura 2012]):*

$$
\begin{aligned}
&\mathcal{E}^{(i)}_{(\alpha)}(u, v) \\
&\quad \equiv \int_S \Big\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}} \, \Phi_\alpha(u, v; y_i, x_i, \mathrm{x} \setminus x_i) \, \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d\mathrm{x}) \\
&\quad = \int_S \int_{\mathbb{R}} \Big\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}} \, \Phi_\alpha(u, v; y_i, x_i, \mathrm{x} \setminus x_i) \\
&\quad \times \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu\big(dx_i \,\big|\, \sigma_{i^c}\big) \, \mu(d\mathrm{x}) \\
&\quad = \int_S \Big\{ \int_{\mathbb{R}^2} I_{\{y_i \neq y'_i\}} \, \Phi_\alpha(u, v; y_i, y'_i, \mathrm{x} \setminus x_i) \\
&\quad \times \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \, \mu\big(dy'_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d\mathrm{x})
\end{aligned}
$$

Theorem 1. *The symmetric non-local forms* $\mathcal{E}_{(\alpha)}$, $0 < \alpha < 2$ *given by (10) (for* $1 < \alpha < 2$ *with the additional assumption (11) with (12)) are (cf. Remark 1-i),ii))*

i) well-defined on $\mathcal{F}C_0^\infty$*;*

ii) Markovian;

iii) closable in $L^2(S;\mu)$*.*

Remark 2. *For $1 < \alpha < 2$ the assumption (11) with (12) can be replaced by the following general one: for each compact $K \subset \mathbb{R}$, there exists an $L_i < \infty$ and*

$$\sup_{y \in \mathbb{R}} I_K(y) \int_{\mathbb{R}} \frac{I_K(y')}{|y - y'|^{2\alpha - 1}} \mu(dy' \mid \sigma_{i^c}) \leq L_i, \quad \mu - a.e..$$

To prove the theorem, we have to show that

i-1) for any Borel measurable u such that $u = 0$, $\mu - a.e.$, it holds that $\mathcal{E}_{(\alpha)}(u, u) = 0$, and

i-2) for any $u, v \in \mathcal{F}C_0^{\infty}$, $\mathcal{E}_{(\alpha)}(u, v) \in \mathbb{R}$,

For the statement ii), we have to show that (cf. [Fukushima]) for any $\epsilon > 0$ there exists a real function $\varphi_\epsilon(t)$, $-\infty < t < \infty$, such that $\varphi_\epsilon(t) = t$, $\forall t \in [0, 1]$, $-\epsilon \leq \varphi_\epsilon(t) \leq 1 + \epsilon$, $\forall t \in (-\infty, \infty)$, and $0 \leq \varphi_\epsilon(t') - \varphi_\epsilon(t) \leq t' - t$ for $t < t'$, such that for any $u \in \mathcal{F}C_0^{\infty}$ it holds that $\varphi_\epsilon(u) \in \mathcal{F}C_0^{\infty}$ and

$$\mathcal{E}_{(\alpha)}(\varphi_\epsilon(u), \varphi_\epsilon(u)) \leq \mathcal{E}_{(\alpha)}(u, u). \tag{13}$$

For the statement iii), we have to show the following: For a sequence $\{u_n\}_{n \in \mathbb{N}}$, $u_n \in \mathcal{F}C_0^\infty$, $n \in \mathbb{N}$, if

$$\lim_{n \to \infty} \|u_n\|_{L^2(S;\mu)} = 0, \tag{14}$$

and

$$\lim_{n,m \to \infty} \mathcal{E}_{(\alpha)}(u_n - u_m, u_n - u_m) = 0, \tag{15}$$

then

$$\lim_{n \to \infty} \mathcal{E}_{(\alpha)}(u_n, u_n) = 0. \tag{16}$$

Proof of i-1): For each $i \in \mathbb{N}$ and any real valued $\mathcal{B}(S)$-measurable function u, note that for each $\epsilon > 0$,

$$I_{\{\epsilon < |x_i - y_i|\}}(y_i)\, I_K(y_i) \Phi_\alpha(u, u; y_i, x_i, \mathrm{x} \setminus x_i)$$

defines a $\mathcal{B}(S \times \mathbb{R})$-measurable function. The function $\Phi_\alpha(u, u; y_i, x_i, \mathrm{x} \setminus x_i)$, is defined by setting $v = u$, $x = x_i$, in (2.8). $\mathcal{B}(S \times \mathbb{R})$ is the Borel σ-field of $S \times \mathbb{R}$. $\mathrm{x} = (x_i, i \in \mathbb{N}) \in S$ and $y_i \in \mathbb{R}$. Then, for any compact subset K of $\mathbb{R}$,
$0 \le I_{\{\epsilon < |x_i - y_i|\}}(y_i)\, I_K(y_i) \Phi_\alpha(u, u; y_i, x_i, \mathrm{x} \setminus x_i)$ converges monotonically to $I_{\{y_i \neq x_i\}}(y_i)\, \Phi_\alpha(u, u; y_i, x_i, \mathrm{x} \setminus x_i)$ as $K \uparrow \mathbb{R}$ and $\epsilon \downarrow 0$, for every $y_i \in \mathbb{R}$, $\mathrm{x} \in S$, and by the Fatou's Lemma, we have

$$
\begin{aligned}
&\int_S \left\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}}(y_i)\, \Phi_\alpha(u,u;y_i,x_i,\mathrm{x}\setminus x_i)\, \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \right\} \\
&\quad \times \mu(d\mathrm{x}) \\
&\quad = \int_S \liminf_{K\uparrow \mathbb{R}} \liminf_{\epsilon \downarrow 0} \Big\{ \int_{\mathbb{R}} I_{\{\epsilon < |x_i - y_i|\}}(y_i)\, I_K(y_i) \\
&\quad \times \Phi_\alpha(u,u;y_i,x_i,\mathrm{x}\setminus x_i)\, \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d\mathrm{x}) \\
&\quad \leq \liminf_{K\uparrow \mathbb{R}} \liminf_{\epsilon \downarrow 0} \int_S \Big\{ \int_{\mathbb{R}} I_{\{\epsilon < |x_i - y_i|\}}(y_i)\, I_K(y_i) \\
&\quad \times \Phi_\alpha(u,u;y_i,x_i,\mathrm{x}\setminus x_i)\, \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d\mathrm{x}), \qquad (17)
\end{aligned}
$$

where K denotes a compact set of $\mathbb{R}$. For any $\epsilon > 0$,

$$\begin{aligned}
&\int_S \left\{ \int_{\mathbb{R}} I_{\{\epsilon < |x_i - y_i|\}}(y_i)\, I_K(y_i)\, \frac{1}{|y_i - x_i|^{2\alpha+1}} \right. \\
&\quad \left. \times \big(u(x_1, \dots)\big)^2 \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \right\} \mu(d\mathrm{x}) \\
&\leq \frac{1}{\epsilon^{2\alpha+1}} \int_S \left\{ \int_{\mathbb{R}} I_{\{\epsilon < |x_i - y_i|\}}(y_i)\, I_K(y_i) \right. \\
&\quad \left. \times \big(u(x_1, \dots)\big)^2 \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \right\} \mu(d\mathrm{x}) \\
&\leq \frac{1}{\epsilon^{2\alpha+1}} \int_S \left\{ \int_{\mathbb{R}} \big(u(x_1, \dots)\big)^2 \right. \\
&\quad \left. \times \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \right\} \mu(d\mathrm{x}) \qquad (18) \\
&= \frac{1}{\epsilon^{2\alpha+1}} \int_S \big(u(x_1, \dots)\big)^2 \mu(d\mathrm{x}),
\end{aligned}$$

also,

$$\int_S \big(u(x_1,\dots)\big)^2 \Big\{ \int_{\mathbb{R}} I_{\{\epsilon<|x_i-y_i|\}}(y_i)\, I_K(y_i) \times \frac{1}{|y_i-x_i|^{2\alpha+1}}\, \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d\mathrm{x}) \leq \frac{1}{\epsilon^{2\alpha+1}} \int_S \big(u(x_1,\dots)\big)^2 \mu(d\mathrm{x}), \tag{19}$$

and from (18), by the Cauchy Schwaz's inequality

$$\Big| \int_S u(x_1,\dots) \Big\{ \int_{\mathbb{R}} I_{\{\epsilon<|x_i-y_i|\}}(y_i)\, I_K(y_i) \times \frac{1}{|y_i-x_i|^{2\alpha+1}} u(x_1,\dots)\, \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d\mathrm{x}) \Big| \leq \frac{1}{\epsilon^{2\alpha+1}} \int_S \big(u(x_1,\dots)\big)^2 \mu(d\mathrm{x}). \tag{20}$$

By (8) and (17), from (18), (19) and (20), for any Borel measurable function u on S such that

$$u(x_1, \dots) = 0, \qquad \mu - a.e.,$$

it holds that

$$\mathcal{E}_{(\alpha)}^{(i)}(u, u) = 0, \quad \forall i \in \mathbb{N}, \qquad \mathcal{E}_{(\alpha)}(u, u) = 0.$$

In order to show i-2), for $0 < \alpha \leq 1$,

take any *representation* $\tilde{u} \in C_0^\infty(\mathbb{R}^n)$ **of** $u \in \mathcal{F}C_0^\infty$, $n \in \mathbb{N}$. **Using** $0 < \alpha + 1 \leq 2$, **it is easy to see from the definition (8) that there exists an** $M < \infty$ **depending on** $\tilde{u}$ **such that**

$$0 \leq \Phi_\alpha(\tilde{u}, \tilde{u}; y_i, y_i', \mathrm{x} \backslash x_i) \leq M, \quad \forall \mathrm{x} \in S, \text{ and } \forall y_i,\ y_i' \in \mathbb{R}. \tag{21}$$

Since, $u = \tilde{u} + \overline{0}$ **for some real valued** $\mathcal{B}(S)$**-measurable function** $\overline{0}$ **such that** $\overline{0} = 0$, μ**-a.e., by (21) together with i-1) and the the Cauchy Schwarz's inequality, for** $u \in \mathcal{F}C_0^\infty$, $\mathcal{E}_{(\alpha)}(u, u) \in \mathbb{R}$, $0 < \alpha \leq 1$, **is identical with** $\mathcal{E}_{(\alpha)}(\tilde{u}, \tilde{u})$ **and well-defined (in fact, for only a finite number of** $i \in \mathbb{N}$. **we have** $\mathcal{E}_{(\alpha)}^{(i)}(u, u) \neq 0$, **cf. also (10)). Then by the Cauchy Schwarz's inequality i-2) follows.**

A proof of ii) is very similar to the one given in section 1 of [Fukushima].

Proof of iii): Suppose that a sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies (14) and (15). Then, by (14) there exists a measurable set $\mathcal{N} \in \mathcal{B}(S)$ and a sub sequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\mu(\mathcal{N}) = 0, \quad \lim_{n_k\to\infty} u_{n_k}(\mathrm{x}) = 0, \quad \forall \mathrm{x} \in S \setminus \mathcal{N}.$$

Define

$$\tilde{u}_{n_k}(\mathrm{x}) = u_{n_k}(\mathrm{x}) \text{ for } \mathrm{x} \in S \setminus \mathcal{N}, \quad \text{and} \quad \tilde{u}_{n_k}(\mathrm{x}) = 0 \text{ for } \mathrm{x} \in \mathcal{N}.$$

Then,

$$\tilde{u}_{n_k}(\mathrm{x}) = u_{n_k}(\mathrm{x}), \ \mu\text{−}a.e., \qquad \lim_{n_k\to\infty} \tilde{u}_{n_k}(\mathrm{x}) = 0, \quad \forall \mathrm{x} \in S. \tag{22}$$

By the fact i-1), for each i,

$$\int_S \Big\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}}(y_i)\, \Phi_\alpha(u_n, u_n; y_i, x_i, \mathrm{x}\backslash x_i)\, \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d\mathrm{x})$$

$$= \int_S \Big\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}}(y_i) \lim_{n_k \to \infty} \Phi_\alpha(u_n - \tilde{u}_{n_k}, u_n - \tilde{u}_{n_k}; y_i, x_i, \mathrm{x}\backslash x_i)$$

$$\times \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d\mathrm{x})$$

$$\leq \liminf_{n_k \to \infty} \int_S \Big\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}}\, \Phi_\alpha(u_n - \tilde{u}_{n_k}, u_n - \tilde{u}_{n_k}; y_i, x_i, \mathrm{x}\backslash x_i)$$

$$\times \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d\mathrm{x})$$

$$= \liminf_{n_k \to \infty} \int_S \Big\{ \int_{\mathbb{R}} I_{\{y_i \neq x_i\}}\, \Phi_\alpha(u_n - u_{n_k}, u_n - u_{n_k}; y_i, x_i, \mathrm{x}\backslash x_i)$$

$$\times \mu\big(dy_i \,\big|\, \sigma_{i^c}\big) \Big\} \mu(d\mathrm{x})$$

$$\equiv \liminf \mathcal{E}^{(i)}_{(\alpha)}(u_n - u_{n_k}, u_n - u_{n_k}). \qquad (23)$$

Now, by using the assumption (15) to the right hand side of (23), we get

$$\lim_{n\to\infty} \mathcal{E}_{(\alpha)}^{(i)}(u_n, u_n) = 0, \qquad \forall i \in \mathbb{N}. \tag{24}$$

(24) together with i) shows that for each $i \in \mathbb{N}$, $\mathcal{E}_{(\alpha)}^{(i)}$ with the domain $\mathcal{F}C_0^{\infty}$ is closable in $L^2(S;\mu)$. Since, $\mathcal{E}_{(\alpha)} \equiv \sum_{i\in\mathbb{N}} \mathcal{E}_{(\alpha)}^{(i)}$, by using the Fatou's Lemma, from (24) and the assumption (15) we see that

$$\mathcal{E}_{(\alpha)}(u_n, u_n) = \sum_{i\in\mathbb{N}} \lim_{m\to\infty} \mathcal{E}_{(\alpha)}^{(i)}(u_n - u_m, u_n - u_m)$$

$$\leq \liminf_{m\to\infty} \mathcal{E}_{(\alpha)}(u_n - u_m, u_n - u_m) \to 0 \text{ as } n \to \infty.$$

This proves (16) (cf. Proposition I-3.7 of [MR] for a general argument). The proof of iii) is completed.

The proof of Theorem 1, for $1 < \alpha < 2$

The proof of i-1), ii) and iii) can be carried out by the completely same manner as the previous proof we have provided for the case $0 < \alpha \leq 1$. We only show that i-2), i.e., $\mathcal{E}_{(\alpha)}(u,u) < \infty$, $\forall u \in \mathcal{F}C_0^\infty$ also holds when we make use of the additional assumption (11) with (12),.
The detailed proof is omitted.

For each $i \in \mathbb{N}$, denote by X_i the random variable that represents the coordinate x_i of $\mathrm{x} = (x_1, x_2, \dots)$:

$$X_i \,: S \ni \mathrm{x} \longmapsto x_i \in \mathbb{R}. \qquad (25)$$

Then

$$\int_S 1_B(x_i)\, \mu(d\mathrm{x}) = \mu(X_i \in B), \quad \text{for} \quad B \in \mathcal{B}(S). \qquad (26)$$

Theorem 2 (Strictly Quasi-regularlity). *Let* $0 < \alpha \leq 1$, *and* $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *be the closed Markovian symmetric form defined through Theorem 1.*

i) In case where $S = l^p_{(\beta_i)}$, $1 \leq p < \infty$, *if there exists positive* l^p *sequence* $\{\gamma_i^{-1}\}_{i \in \mathbb{N}}$ *(for e.g.,* $\gamma_i = i^{\frac{1+\delta}{p}}$ *for some* $\delta > 0$*), and an* $M_0 < \infty$ *and*

$$\sum_{i=1}^{\infty} \beta_i^{\frac{2}{p}} \gamma_i^2 \cdot \mu\Big(|X_i| > M_0 \cdot \beta_i^{-\frac{1}{p}} \gamma_i^{-1} \Big) < \infty, \qquad (27)$$

$$\mu\Big(\bigcup_{M \in \mathbb{N}} \{ |X_i| \leq M \cdot \beta_i^{-\frac{1}{p}} \gamma_i^{-\frac{1}{p}}, \forall i \in \mathbb{N} \} \Big) = 1, \qquad (28)$$

hold, then $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *is a strictly quasi-regular Dirichlet form.*

ii) In case where $S = l^{\infty}_{(\beta_i)}$ *defined by (2), if there exist an* $M_0 < \infty$ *and a sequence* $\{\gamma_i\}_{i \in \mathbb{N}}$ *such that* $0 < \gamma_1 \leq \gamma_2 \leq \cdots \to \infty$*, and both*

$$\sum_{i=1}^{\infty} \beta_i^2 \gamma_i^2 \cdot \mu\Big(|X_i| > M_0 \cdot \beta_i^{-1} \gamma_i^{-1}\Big) < \infty, \tag{29}$$

$$\mu\Big(\bigcup_{M \in \mathbb{N}} \{|X_i| \leq M \cdot \beta_i^{-1} \gamma_i^{-1}, \forall i \in \mathbb{N}\}\Big) = 1, \tag{30}$$

hold, then $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *is a strictly quasi-regular Dirichlet form.*

iii) In case where $S = \mathbb{R}^{\mathbb{N}}$ *defined by (3),* $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *is a strictly quasi-regular Dirichlet form.*

Proof of Theorem 2. We have to show that $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ satisfies

i) There exists an $\mathcal{E}_{(\alpha)}$-nest $(D_M)_{M \in \mathbb{N}}$ consisting of compact sets.

ii) There exists a subset of $\mathcal{D}(\mathcal{E}_{(\alpha)})$, that is dense with respect to the norm $\| \cdot \|_{L^2(S;\mu)} + \sqrt{\mathcal{E}_{(\alpha)}}$. And the elements of the subset have $\mathcal{E}_{(\alpha)}$-quasi continuous versions.

iii) There exists $u_n \in \mathcal{D}(\mathcal{E}_{(\alpha)})$, $n \in \mathbb{N}$, having $\mathcal{E}_{(\alpha)}$-quasi continuous μ-versions $\tilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}_{(\alpha)}$-exceptional set $\mathcal{N} \subset S$ such that $\{\tilde{u}_n \,:\, n \in \mathbb{N}\}$ separates the points of $S \setminus \mathcal{N}$.

iv) For the *strictly* quasi-regularity, it suffices to show that $1 \in \mathcal{D}(\mathcal{E}_{(\alpha)})$.

For the case where $S = l^p_{(\beta_i)}$, for simplicity, let $\gamma_i^{-1} = i^{-\frac{1+\delta}{p}}$ for some $\delta > 0$. A key point of the proof is the fact that for each $M \in \mathbb{N}$,

$$D_M \equiv \left\{ \mathrm{x} \in l^p_{(\beta_i)} \,:\, \beta_i^{\frac{1}{p}} |x_i| \le M \cdot i^{-\frac{1+\delta}{p}},\, i \in \mathbb{N} \right\}, \quad (31)$$

is a compact set in $S = l^p_{(\beta_i)}$.

Note that D_M is not identical to but a proper subset of the bounded set such that

$$\left\{ \mathrm{x} \in l^p_{(\beta_i)} \,:\, \left(\sum \beta_i |x_i|^p \right)^{\frac{1}{p}} \le M \right\}.$$

Let $\eta(\cdot) \in C_0^\infty(\mathbb{R})$ be a function such that $\eta(x) \geq 0$, $|\frac{d}{dx}\eta(x)| \leq 1, \quad \forall x \in \mathbb{R}$ and

$$\eta(x) = \begin{cases} 1, & |x| \leq 1; \\ 0, & |x| \geq 3. \end{cases} \tag{32}$$

For each $M \in \mathbb{N}$ and $i \in \mathbb{N}$, let

$$\eta_{M,i}(x) \equiv \eta\left(M^{-1} \cdot i^{\frac{1+\delta}{p}} \beta_i^{\frac{1}{p}} \cdot x\right), \quad x \in \mathbb{R},$$

then, $\prod_{i \geq 1} \eta_{M,i} \in l^p_{(\beta_i)}$, $\operatorname{supp}\left[\prod_{i\geq 1} \eta_{M,i}\right] \subset D_{3M}$, $M \in \mathbb{N}$.

For each $f \in C_0^\infty(\mathbb{R}^n \to \mathbb{R})$, $n \in \mathbb{N}$, define

$$f_M(x_1, \ldots, x_n, x_{n+1}, \ldots) \equiv f(x_1, \ldots, x_n) \cdot \prod_{i \geq 1} \eta_{M,i}(x_i). \tag{33}$$

Under the condition (27), it is possible to show that $f_M \in \mathcal{D}(\mathcal{E}_{(\alpha)})$. Also, by (28), it is possible to show that there exists a subsequence $\{f_{M_l}\}_{l \in \mathbb{N}}$ of $\{f_M\}_{M \in \mathbb{N}}$ such that the *Cesaro mean*

$$w_m \equiv \frac{1}{m} \sum_{l=1}^{m} f_{M_l} \to u \equiv f \cdot \prod_{i \geq 1} 1_{\mathbb{R}}(x_i)$$

$$\text{in} \quad \mathcal{D}(\mathcal{E}_{(\alpha)}) \quad \text{as} \quad n \to \infty. \tag{34}$$

(34) shows that

the linear hull of $\Big\{ f_M,\ M \in \mathbb{N} \ : \ f \in C_0^{\infty}(\mathbb{R}^n \to \mathbb{R}),\ n \in \mathbb{N} \Big\}$.

can be taken as an $\mathcal{D}(\mathcal{E}_{(\alpha)})$-nest.

Theorem 3 (Strictly Quasi-regularlity). *Let* $1 < \alpha < 2$. *Suppose that the assumption (11) with (12) hold. Let* $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *be the closed Markovian symmetric form defined at the beginning of this section through Theorem 1. Then the following statements hold:*

i) In the case where $S = l^p_{(\beta_i)}$, $1 \le p < \infty$, *as defined by (1), if there exists a positive* l^p *sequence* $\{\gamma_i^{-\frac{1}{p}}\}_{i \in \mathbb{N}}$ *and an* $M_0 < \infty$, *and both (28),*

$$\sum_{i=1}^{\infty} (\beta_i^{\frac{1}{p}} \gamma_i^{\frac{1}{p}})^{\alpha+1} \cdot \mu\Big(\beta_i^{\frac{1}{p}} |X_i| > M_0 \cdot \gamma_i^{-\frac{1}{p}}\Big) < \infty, \tag{35}$$

$$\lim_{M \to \infty} M^{-\alpha} \sum_{i=1}^{\infty} L_{M,i} \cdot (\beta_i^{\frac{1}{p}} \gamma_i^{\frac{1}{p}})^{\alpha} \cdot \mu\Big(\beta_i^{\frac{1}{P}} |X_i| > M \cdot \gamma_i^{-\frac{1}{p}}\Big) < \infty, \tag{36}$$

hold, then $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *is a strictly quasi-regular Dirichlet form, where for each* $M \in \mathbb{N}$ *and* $i \in \mathbb{N}$, $L_{M,i}$ *is the bound of the conditional probability density* ρ *for a given compact set*

$$K_{M,i} \equiv \big[- 6M \cdot \beta_i^{-\frac{1}{p}} \gamma_i^{-\frac{1}{p}}, 6M \cdot \beta_i^{-\frac{1}{p}} \gamma_i^{-\frac{1}{p}} \big] \subset \mathbb{R}$$

in the assumption (12).

ii) In the case where $S = l^{\infty}_{(\beta_i)}$ *as defined by (2), if there exists a sequence* $\{\gamma_i\}_{i\in\mathbb{N}}$ *such that* $0 < \gamma_1 \leq \gamma_2 \leq \cdots \to \infty$*, and both*

$$\sum_{i=1}^{\infty} \big(\beta_i\gamma_i\big)^{\alpha+1} \cdot \mu\Big(\beta_i|X_i| > M_0 \cdot \gamma_i^{-1}\Big) < \infty, \quad \textit{for some } M_0 < \infty, \tag{37}$$

$$\lim_{M\to\infty} M^{-\alpha} \sum_{i=1}^{\infty} L_{M,i} \cdot \big(\beta_i\gamma_i\big)^{\alpha} \cdot \mu\Big(\beta_i|X_i| > M \cdot \gamma_i^{-1}\Big) < \infty, \tag{38}$$

and (30) hold, then $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *is a strictly quasi-regular Dirichlet form, where for each* $M \in \mathbb{N}$ *and* $i \in \mathbb{N}$, $L_{M,i}$ *is the bound of the conditional probability density* ρ *for a given compact set*

$$K_{M,i} \equiv \big[-6M \cdot \beta_i^{-1}\gamma_i^{-1}, 6M \cdot \beta_i^{-1}\gamma_i^{-1}\big] \subset \mathbb{R}$$

iii) *In the case where* $S = \mathbb{R}^{\mathbb{N}}$ *as defined by (2.3), if there exists a sequence* $\{\gamma_i\}_{i \in \mathbb{N}}$ *such that* $0 < \gamma_i$, $\forall i \in \mathbb{N}$, *and*

$$\lim_{M \to \infty} M^{-\alpha} \sum_{i=1}^{\infty} L_{M,i} \cdot \gamma_i^{-\alpha} \cdot \mu\Big(|X_i| > M \cdot \gamma_i\Big) < \infty, \tag{39}$$

holds, then $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *is a strictly quasi-regular Dirichlet form, where for each* $M \in \mathbb{N}$ *and* $i \in \mathbb{N}$, $L_{M,i}$ *is the bound of the conditional probability density* ρ *for a given compact set*

$$K_{M,i} \equiv \big[-6M \cdot \gamma_i, \, 6M \cdot \gamma_i \big] \subset \mathbb{R}$$

in the assumption (2.12).

Theorem 4. *Let* $0 < \alpha < 2$*, and let* $\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$ *be a strictly quasi-regular Dirichlet form on* $L^2(S; \mu)$ *that is defined through Theorem 2 or Theorem 3. Then for* $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$*, there exists a properly associated* μ*-tight special standard process, in short a strong Markov process taking values in* S *and having right continuous trajectories with left limits up to the life time (cf. Definitions IV-1.5, 1.8 and 1.13 of [MR] for its precise definition),*

$$\mathbb{M} \equiv \Big(\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_{\mathbf{x}})_{\mathbf{x} \in S_\triangle}\Big),$$

where $\triangle$ *is an adjoined extra points, called as the cemetery, of* S*.*

Euclidean (*scalar*) quantum fields are expressed as random fields on $\mathcal{S}' \equiv \mathcal{S}'(\mathbb{R}^d \to \mathbb{R})$, or, resp., $\mathcal{S}'(\mathbb{T}^d \to \mathbb{R})$, the Schwartz's space of real tempered distributions on $\mathbb{R}^d$, resp., the d-dimensional torus $\mathbb{T}^d$, with $d \geq 1$ a given space time dimension. Hence, each Euclidean quantum field is taken as a probability space $\left(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \nu\right)$, where $\mathcal{B}(\mathcal{S}')$ is the Borel σ-field of $\mathcal{S}'$ and ν is a Borel probability measure on $\mathcal{S}'$. One of the standard theorem through which such ν are constructed is the Bochner-Minlos's theorem (cf. e.g., Section 3.2 of [Hida]), which is an existence theorem of probability measures on Hilbert nuclear spaces. Since the space $\mathcal{S}$ and its dual $\mathcal{S}'$ is a Hilbert nuclear space, and by making use of a Hilbert-Schmidt operators defined on it, we can adapt our Theorems.

Let

$$\mathcal{H}_0 \equiv \Big\{ f \,:\, \|f\|_{\mathcal{H}_0} = \big((f,f)_{\mathcal{H}_0}\big)^{\frac{1}{2}} < \infty,\ f:\mathbb{R}^d \to \mathbb{R},$$

$$\text{measurable}\Big\} \supset \mathcal{S}(\mathbb{R}^d), \tag{40}$$

where

$$(f,g)_{\mathcal{H}_0} \equiv (f,g)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(x)g(x)\,dx. \tag{41}$$

Let

$$H \equiv (|x|^2+1)^{\frac{d+1}{2}}(-\Delta+1)^{\frac{d+1}{2}}(|x|^2+1)^{\frac{d+1}{2}}, \tag{42}$$

$$H^{-1} \equiv (|x|^2+1)^{-\frac{d+1}{2}}(-\Delta+1)^{-\frac{d+1}{2}}(|x|^2+1)^{-\frac{d+1}{2}}, \tag{43}$$

be the pseudo differential operators on $\mathcal{S}'(\mathbb{R}^d \to \mathbb{R}) \equiv \mathcal{S}'(\mathbb{R}^d)$ with the d-dimensional Laplace operator Δ.

For each $n \in \mathbb{N}$, define

$\mathcal{H}_n \equiv$ the completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to the norm

$$\|f\|_n = \sqrt{(f,f)_n} \text{ with } (f,g)_n = (H^n f, H^n g)_{\mathcal{H}_0}, \quad (44)$$

and

$\mathcal{H}_{-n} \equiv$ the completion of $\mathcal{S}'(\mathbb{R}^d)$ with respect to the norm

$$\|f\|_{-n} = \sqrt{(f,f)_{-n}} \text{ with } (f,g)_{-n} = ((H^{-1})^n f, (H^{-1})^n g)_{\mathcal{H}_0}. \quad (45)$$

by taking an inductive limit $\mathcal{H} = \bigcap_{n\in\mathbb{N}} \mathcal{H}_n$, then

$$\mathcal{H} \subset \cdots \subset \mathcal{H}_{n+1} \subset \mathcal{H}_n \subset \cdots \subset \mathcal{H}_0 \subset \cdots \subset \mathcal{H}_{-n} \subset \mathcal{H}_{-n-1} \subset \cdots \subset \mathcal{H}^*. \quad (46)$$

For the positive self-adjoint operator H^{-1} on $\mathcal{H}_0 = L^2(\mathbb{R}^d \to \mathbb{R})$,take the orthonormal base (O.N.B.) $\{\varphi_i\}_{i\in\mathbb{N}}$ of $\mathcal{H}_0$ such that

$$H^{-1}\varphi_i = \lambda_i \varphi_i, \qquad i \in \mathbb{N}, \tag{47}$$

where $\{\lambda_i\}_{i\in\mathbb{N}}$ is the corresponding eigenvalues such that $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots > 0$, which satisfies

$$\sum_{i\in\mathbb{N}} (\lambda_i)^2 < \infty, \quad \text{i.e.,} \quad \{\lambda_i\}_{i\in\mathbb{N}} \in l^2. \tag{48}$$

Then,

$$\{(\lambda_i)^n \varphi_i\}_{i\in\mathbb{N}} \quad \text{is an O.N.B. of } \mathcal{H}_n \tag{49}$$

and

$$\{(\lambda_i)^{-n} \varphi_i\}_{i\in\mathbb{N}} \quad \text{is an O.N.B. of } \mathcal{H}_{-n} \tag{50}$$

Thus, by the Fourier series expansion for $f \in \mathcal{H}_m$,

$$f = \sum_{i \in \mathbb{N}} a_i (\lambda_i^m \varphi_i), \quad \text{with}$$

$$a_i \equiv \big(f,\, (\lambda_i^m \varphi_i)\big)_m = \lambda_i^{-m} (f,\, \varphi_i)_{L^2},\ i \in \mathbb{N}, \tag{51}$$

we have an *isometric isomorphism* τ_m for each $m \in \mathbb{Z}$ such that

$$\tau_m \,:\, \mathcal{H}_m \ni f \longmapsto (\lambda_1^m a_1, \lambda_2^m a_2, \dots) \in l^2_{(\lambda_i^{-2m})}, \tag{52}$$

where $l^2_{(\lambda_i^{-2m})}$ is the weighted l^2 space defined by (1) with $p = 2$, and $\beta_i = \lambda_i^{-2m}$.

$$\mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_0 = L^2(\mathbb{R}^d) \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2}, \tag{53}$$

$$l^2_{(\lambda_i^{-4})} \subset l^2_{(\lambda_i^{-2})} \subset l^2 \subset l^2_{(\lambda_i^{2})} \subset l^2_{(\lambda_i^{4})}. \tag{54}$$

Example 1. (The Euclidean free fields)

Let ν_0 be the Euclidean free field measure on $\mathcal{S}' \equiv \mathcal{S}'(\mathbb{R}^d)$, precisely, the the corresponding (generalized) characteristic function, in the sense the Bochner Minlos's theorem, $C(\varphi) \equiv \int_{\mathcal{S}'} e^{i<\phi,\varphi>} \nu_0(d\phi)$ is given by

$$C(\varphi) = \exp(-\frac{1}{2}(\varphi, (-\Delta + m_0^2)^{-1}\varphi)_{L^2(\mathbb{R}^d)}), \ \varphi \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R}), \tag{55}$$

Equivalently, ν_0 is a centered Gaussian probability measure on $\mathcal{S}'$, the covariance of which is , for φ_1, $\varphi_2 \in \mathcal{S}(\mathbb{R}^d \to \mathbb{R})$,

$$\int_{\mathcal{S}'} < \phi, \varphi_1 > \cdot < \phi, \varphi_1 > \ \nu_0(d\phi) = (\varphi_1, (-\Delta + m_0^2)^{-1}\varphi_2), \tag{56}$$

where Δ is the d-dimensional Laplace operator and $m_0 > 0$.

By (55), the functional $C(\varphi)$ is continuous with respect to the norm of the space $\mathcal{H}_0 = L^2(\mathbb{R}^d)$, and the kernel of $(-\Delta + m_0^2)^{-1}$, which is the Fourier inverse transform of $(|\xi|^2 + m_0^2)^{-1}$, $\xi \in \mathbb{R}^d$, is explicitly given by Bessel functions. By the Bochner Minlos's theorem, the support of ν_0 can be taken to be in the wider Hilbert spaces $\mathcal{H}_{-n}$, $n \geq 1$. We take ν_0 as a Borel probability measure on $\mathcal{H}_{-2}$. By (52), by taking $m = -2$, τ_{-2} defines an isometric isomorphism such that

$$\tau_{-2} : \mathcal{H}_{-2} \ni f \longmapsto (a_1, a_2, \dots) \in l^2_{(\lambda_i^4)}, \tag{57}$$

$$\text{with} \quad a_i \equiv (f, \lambda_i^{-2} \varphi_i)_{-2}, \;\; i \in \mathbb{N}.$$

Define a probability measure μ on $l^2_{(\lambda_i^4)}$ such that

$$\mu(B) \equiv \nu_0 \circ \tau_{-2}^{-1}(B) \quad \text{for} \quad B \in \mathcal{B}(l^2_{(\lambda_i^4)}).$$

We set $S = l^2_{(\lambda_i^4)}$ in Theorems 2 and 4, then it follows that the weight β_i satisfies $\beta_i = \lambda_i^4$. We can take ${\gamma_i}^{-\frac{1}{2}} = \lambda_i$ in Theorem 2-i) with $p = 2$, then, from (48) we have

$$\sum_{i=1}^{\infty} \beta_i \gamma_i \cdot \mu\Big(\beta_i^{\frac{1}{2}} |X_i| > M \cdot \gamma_i^{-\frac{1}{2}}\Big) \leq \sum_{i=1}^{\infty} \beta_i \gamma_i = \sum_{i=1}^{\infty} (\lambda_i)^2 < \infty \tag{58}$$

(58) shows that the condition (27) holds.

Also, as has been mentioned above, since $\nu_0(\mathcal{H}_{-n}) = 1$, for any $n \geq 1$, we have

$$1 = \nu_0(\mathcal{H}_{-1}) = \mu(l^2_{(\lambda_i^2)}) = \mu\Big(\bigcup_{M \in \mathbb{N}} \{|X_i| \leq M \beta_i^{-\frac{1}{2}} \gamma_i^{-\frac{1}{2}}, \forall i \in \mathbb{N}\}\Big)$$

$$\text{for } \beta_i = \lambda_i^4, \quad \gamma_i^{-\frac{1}{2}} = \lambda_i.$$

This shows that the condition (28) is satisfied.

Thus, by Theorem 2-i) and Theorem 4, for each $0 < \alpha \leq 1$, there exists an $l^2_{((\lambda_i)^4)}$-valued Hunt process $\mathbb{M} \equiv \big(\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_{\mathrm{x}})_{\mathrm{x} \in S_\triangle}\big)$, associated to the non-local Dirichlet form $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$.

We can define an $\mathcal{H}_{-2}$-valued process $(Y_t)_{t\geq 0}$ such that

$$(Y_t)_{t\geq 0} \equiv \left(\tau_{-2}^{-1}(X_t)\right)_{t\geq 0}.$$

Equivalently, by (57) for $X_t = (X_1(t), X_2(t), \dots) \in l^2_{(\lambda_i^4)}$, $P_{\mathrm{x}} - a.e.$, by setting $A_i(t)$ such that $A_i(t) \equiv \lambda_i X_i(t)$, we see that Y_t is also given by

$$Y_t = \sum_{i\in\mathbb{N}} A_i(t)(\lambda_i^{-2}\varphi_i) = \sum_{i\in\mathbb{N}} X_i(t)\varphi_i \in \mathcal{H}_{-2},\ \forall t \geq 0,\ \ P_{\mathrm{x}}-a.e.. \tag{59}$$

Y_t is an $\mathcal{H}_{-2}$-valued Hunt process that is a *stochastic quantization* with respect to the non-local Dirichlet form $(\tilde{\mathcal{E}}_{(\alpha)}, \mathcal{D}(\tilde{\mathcal{E}}_{(\alpha)}))$ on $L^2(\mathcal{H}_{-2}, \nu_0)$, that is defined through $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, by making use of τ_{-2}. This holds for all $0 < \alpha \leq 1$.

Example 2. (The Euclidean Φ_3^4 fields)

By making use of the results in [Brydges,Fröhlich,Sokal 83], through the Bochner-Minlos's theorem, the prorability measure ν of the Φ_3^4 Euclidean field on $\mathbb{R}^3$ has the (generalized) characteristic function

$$C(\varphi) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \langle S_{2n}, \varphi^{\otimes 2n} \rangle. \tag{60}$$

It is possible to show that

$$|C(\varphi) - 1| \leq e^{\frac{1}{2} K \|\varphi\|_{\mathcal{H}_1}} - 1, \qquad \forall \varphi \in \mathcal{H}_1. \tag{61}$$

Thus, for any $n \geq 2$

$$\nu \text{ is the probability measure on } \mathcal{H}_{-n} \text{ corresponding to } C(\varphi). \tag{62}$$

By taking $n = -3$, τ_{-3} defines an isometric isomorphism such that

$$\tau_{-3} : \mathcal{H}_{-3} \ni f \longmapsto (a_1, a_2, \dots) \in l^2_{(\lambda_i^6)}, \quad \text{with}$$

$$a_i \equiv (f, \lambda_i^{-3} \varphi_i)_{-3}, \ \ i \in \mathbb{N}. \tag{63}$$

Define μ on $l^2_{(\lambda_i^6)}$ such that

$$\mu(B) \equiv \nu \circ \tau_{-3}^{-1}(B) \quad \text{for} \quad B \in \mathcal{B}(l^2_{(\lambda_i^6)}). \tag{64}$$

Set $S = l^2_{(\lambda_i^6)}$. We can take $\beta_i = \lambda_i^6$, ${\gamma_i}^{-\frac{1}{2}} = \lambda_i$ in Theorem 2-i) with $p = 2$, then,

$$\sum_{i=1}^{\infty} \beta_i \gamma_i \cdot \mu\Big(\beta_i^{\frac{1}{2}} |X_i| > M \cdot {\gamma_i}^{-\frac{1}{2}}\Big) \leq \sum_{i=1}^{\infty} \beta_i \gamma_i = \sum_{i=1}^{\infty} (\lambda_i)^4 < \infty. \tag{65}$$

This shows that (27) holds, also, it is possible to see that (28) holds.
Thus, by Theorem 2-i) and Theorem 4, for each $0 < \alpha \leq 1$, there exists an $l^2_{(\lambda_i^6)}$-valued Hunt process

$$\mathbb{M} \equiv \left(\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_{\mathrm{x}})_{\mathrm{x} \in S_\triangle}\right), \qquad (66)$$

associated to the non-local Dirichlet form $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$.
Then define an $\mathcal{H}_{-3}$-valued process $(Y_t)_{t \geq 0}$ such that $(Y_t)_{t \geq 0} \equiv \left(\tau_{-2}^{-1}(X_t)\right)_{t \geq 0}$.

Equivalently, by (44) for $X_t = (X_1(t), X_2(t), \dots) \in l^2_{(\lambda_i^6)}$, $P_{\mathrm{x}} - a.e.$, by setting $A_i(t)$ such that $X_i(t) = \lambda_i^{-3} A_i(t)$, then Y_t is given by

$$Y_t = \sum_{i \in \mathbb{N}} A_i(t)(\lambda_i^{-3}\varphi_i) = \sum_{i \in \mathbb{N}} X_i(t)\varphi_i \in \mathcal{H}_{-3},$$

$$\forall t \geq 0, \ P_{\mathrm{x}} - a.e.. \qquad (67)$$

It is an $\mathcal{H}_{-3}$-valued Hunt process that is a *stochastic quantization* **with respect to the non-local Dirichlet form $(\tilde{\mathcal{E}}_{(\alpha)}, \mathcal{D}(\tilde{\mathcal{E}}_{(\alpha)}))$ on $L^2(\mathcal{H}_{-3}, \nu)$, that is defined through $(\mathcal{E}_{(\alpha)}, \mathcal{D}(\mathcal{E}_{(\alpha)}))$, by making use of τ_{-3}.**

Example 1. (The Høegh-Krohn model with $d = 2$)

For $|a_0| < \sqrt{4\pi}$ and $g \in L^2(\mathbb{R}^2 \to \mathbb{R}) \cap L^1(\mathbb{R}^2 \to \mathbb{R})$, on the measure space $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \nu_0)$, define a random variable

$$V_{exp}(\phi) \equiv \sum_{n=0}^{\infty} \frac{(a_0)^n}{n!} < g, :\phi^n:>, \tag{68}$$

and define a probability measure ν_{exp} on $\mathcal{S}'$ such that

$$\nu_{exp}(d\phi) \equiv Z^{-1} e^{-V_{exp}(\phi)} \nu_0(d\phi), \tag{69}$$

where ν_0 is the 2-dimensional Euclidean free field measure and $\mathcal{S}' \equiv \mathcal{S}'(\mathbb{R}^2 \to \mathbb{R})$, Z is the normalizing constant.

It is known that (cf., e.g., [A,H72], [Simon])

$$V_{exp} \in \cap_{r\geq 1} L^r(\mathcal{S}', \nu_0), \;\; V_{exp}(\phi) \geq 0, \; \nu_0 - a.e., \tag{70}$$

$$0 \leq e^{-V_{exp}(\phi)} \leq 1, \quad \nu_0 - a.e.. \tag{71}$$

Through simple calculations, by making use of the Hölder's inequality, and the Gaussian inequality, it is possible to see that for $C_{exp}(\varphi)$, the characteristic function of ν_{exp},

$$|C_{exp}(\varphi) - 1| \leq Z^{-1}\{(e^{\frac{1}{2}|\varphi|^2} - 1)|\varphi|^2 + |\varphi|^2\}, \; \forall \varphi \in \mathcal{S}', \tag{72}$$

where $|\varphi|^2 \equiv ((-\Delta + 1)\varphi, \varphi)_{L^2}$.

(72) shows that the characteristic function of ν_{exp} possesses the same continuity (in a neighbourhood of the origin) as the one of the Euclidean free field (cf. Example 1). Hence, through the same arguments as were done in the previous examples, for the random field $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \nu_{exp})$ the same results on the *non-local* type stochastic quantizations as the one for the Euclidean free field with $d = 2$ holds.

Example 1. (The $P(\phi)_2$ and the Albeverio, Høegh-Krohn trigonometric model with $d = 2$)

For the 2-dimensional, $(d = 2)$, Euclidean fields with the (truncated) potential term $P(\phi)_2$ and the Albeverio, Høegh-Krohn trigonometric functions, passing through the similar arguments as were performed in the previous examples, with a little indirect way, we see that theses fields can be treated same as the Euclidean free field with $d = 2$.

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