Asymptotic behavior of reflected SPDEs with singular potentials driven by additive noises[1]

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[1]Partially based on the joint work with L. Goudenège

Outline

Random obstacle problems

Reflected stochastic heat equation (RSHE)

The reflected stochastic heat equation driven by the space-time white noise:

$$\begin{cases} \frac{\partial u}{\partial t}(t,\theta) = \frac{1}{2}\frac{\partial^2 u}{\partial \theta^2}(t,\theta) + b(u(t,\theta)) + \dot{W}(t,\theta) + \eta(dtd\theta), \\ u(t,0) = u(t,1) = 0,\ t \geq 0, \\ u(0,x) = h(\theta) \geq 0,\ \theta \in (0,1), \\ u(t,\theta) \geq 0,\ t \geq 0, \theta \in [0,1]\ a.s. \end{cases} \tag{1.1}$$

- $\{\dot{W}(t,\theta) : t \geq 0, \theta \in [0,1]\}$ denotes the space-time white noise.
- η denotes the positive random reflecting measure on $[0,\infty) \times [0,1]$, which forces $u(t,\theta)$ to be nonnegative.

Random obstacle problems

Stochastic Cahn-Hilliard equation with logarithmic free energy (SCH)

Let f denote the logarithmic type nonlinearity

$$f(u) = \begin{cases} +\infty, & u \le -1, \\ \log\left(\dfrac{1-u}{1+u}\right) + \lambda u, & u \in (-1,1), \\ -\infty, & u \ge 1. \end{cases}$$

F denotes the primitive function of $-f$ with $F(0) = 0$, that is,

$$F(u) = (1+u)\log(1+u) + (1-u)\log(1-u) - \frac{\lambda}{2}u^2, u \in (-1,1).$$

SCH

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\theta) \\ = -\dfrac{1}{2}\dfrac{\partial^2}{\partial\theta^2}\left(\dfrac{\partial^2 u}{\partial\theta^2}(t,\theta) + f(u(t,\theta)) + \eta_-(t,\theta) - \eta_+(t,\theta)\right) \\ \qquad + B\dot{W}(t,\theta), \quad t>0,\ \theta\in(0,1), \\ u'(t,0) = u'(t,1) = \dfrac{\partial^3 u}{\partial x^3}(t,0) = \dfrac{\partial^3 u}{\partial x^3}(t,1) = 0,\ t\geq 0, \\ \displaystyle\int_0^1\int_0^1 (1+u(t,\theta))\eta_-(dtd\theta) \\ = \displaystyle\int_0^1\int_0^1 (1-u(t,\theta))\eta_+(dtd\theta) = 0, \\ u(0,\theta) = x(\theta),\ \theta\in(0,1),\ u(t,x)\in[-1,1]\ a.s., \end{cases} \tag{1.2}$$

where η_-, η_+ are two non-negative random measures and B denotes two kinds of operators which will be specified later.

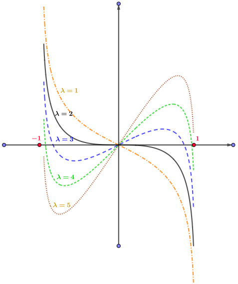

Figure: Graph of f.

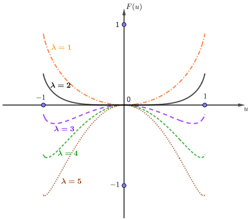

Figure: Graph of F: Double-well potential whenever $\lambda > 2$.

Consider the formal Hamiltonian called the logarithmic free energy or (Ginzburg-Landau free energy)

$$\mathcal{H}(u) = \int_0^1 \left(\frac{1}{2} |\nabla u(\theta)|^2 + F(u(\theta)) \right) d\theta.$$

The SPDEs (1.1) can be formally written as

$$\frac{\partial u}{\partial t}(t,\theta) = -\frac{1}{2}(-\Delta)^{\gamma}\left(\frac{\delta \mathcal{H}(u)}{\delta u(\theta)}(t,\theta) + \xi(t,\theta)\right) + (-\Delta)^{\gamma/2} B\dot{W}(t,\theta). \tag{1.3}$$

where $\xi(t,\theta)$ is a signed random measure, $\frac{\delta \mathcal{H}}{\delta u(\theta)}(u)$ denotes the functional derivative of $\mathcal{H}$ and equals to $-\Delta u(\theta) + F'(u(\theta))$.

Remark

(1) If $\xi = 0$ (and f is "good"), then (1.3) *is called the time-dependent Ginzburg-Landau equation. According to Hohenberg and Halperin (1977), it is called the* **Model A** *(non-conservative one) for $\gamma = 0$ and the* **Model B** *for (conservative one, $\int_0^1 u(t,\theta)d\theta$) for $\gamma = 1$.*
(2) SPDEs (1.1) *and* (1.2) *are called the Model A with reflection and respectively Model B with reflection.*

Remark

One dimensional Skorokhod equation:

$$\begin{cases} dB(t) = dw(t) + d\ell(t),\ t \geq 0, & B(0) = y,\ y \geq 0, \\ B(t) \geq 0, \quad \int_0^\infty B(s)\, d\ell(s) = 0, & \ell(t) \nearrow,\ \ell(0) = 0. \end{cases}$$

Hence, the reflected SPDE (1.1) *and* (1.2) *are usually regarded as an infinite-dimensional Skorokhod problem.*
They are the special cases of the random parabolic obstacle problems, see LNM, 2181(Zambotti, 2017).
Stochastic heat equation, stochastic Cahn-Hilliard equation, stochastic Porous media equation, stochastic Burgers equation, $\cdots$.

Main Topic

To study the ergodic property by establishing dimension-free Harnack inequalities for the Markov simegroups relative to (1.1) and (1.2).

Prior Research of RSHE

- The case of additive noise on bounded domain
  - Nualart and Pardoux (1992): Existence and uniqueness.
  - Funaki and Olla (2001), Zambotti (2001): Reversible measure.
  - Zambotti: Occupation densities (2004), Hitting property (2006).
- Other cases
  - Multiplicative noises: Donati-Martin (1993) (Existence); Xu and Zhang (2009, Uniqueness and LDP); Zhang et al. (2010, Strong Feller property)
  - Double reflections: Otobe (2006), Zhang et al. (2011)$\cdots$.

Prior Research of SCH

- Cahn and Hilliard (1958): The phase separation in a binary alloy.
- Material science, tumor growth, thin films, population dynamics, $\cdots$.
- For thermal fluctuation(Cook, 1970), the noise is required.
- Da Prato, Debussche (1996), Cardon-Weber (2001), Antonopoulou, G. Karali, A. Millet (2016): polynomial f and no restriction on $u(t,\theta)$.

Prior Research of SCH

- One reflection (Comparison theorem fails): Debussche and Zambotti (2007) for $f=0$; Goudenège (2009) for $f=u^{-\alpha}$ or $-\log u$.
- In applications, the solution of the Cahn-Hilliard equation is explained as the rescaled density of atoms or concentration of one of material's components taking values in $[-1,1]$. But, different from the deterministic case, f is too weak.
- Debussche and Goudenège (2011): (1.2) with $B=\frac{d}{d\theta}$.
- Goudenège and Manca (2015): (1.2) with the H-S operator B.

Applications

The fluctuations for $\nabla\phi$ interface models on a hard wall.

- Funaki and Olla (2001): without conservation of the area between the interface and the wall.
- Zambotti (2008): with conservation.

Definition (Dimension-free Harnack inequalities)

- *Let $(P_t)_{t\geq 0}$ be a Markov semigroup on (E,d). We say that P_t satisfies a classical dimension-free Harnack inequality is*

$$\psi(P_t\Phi(x)) \leq P_t(\psi(\Phi)(y)) \exp\{\Psi(t,x,y)\},\ 0 \leq \Phi \in B_b(E), \quad (1.4)$$

where $\psi : [0,\infty) \to [0,\infty)$ is convex, Ψ is a non-negative function defined on $[0,\infty)\times E \times E$ with $\Psi(t,x,x) = 0$.

- *In particular,*
  - *When $\psi(r) = r^p,\ p > 1$, (1.4) is called the Harnack inequality with power p, i.e.,*

$$|P_t\Phi(x)|^p \leq P_t|\Phi|^p(y) \exp\{\Psi(t,x,y)\}$$

  - *When $\psi(r) = e^r$, (1.4) is called the log-Harnack inequality(Log-HI). It is equivalent to*

$$P_t \log \Phi(x) \leq \log P_t\Phi(y) + \Psi(t,x,y), 0 \leq \Phi \in B_b(E), t > 0.$$

- *Harnack inequality with power $p \Longrightarrow$ Log-Harnack inequality.*

Dimension-free Harnack inequalities

For a function ϕ on E, we denote by $|\nabla\phi|(x)$ its local Lipschitz constant at x, that is,

$$|\nabla\phi|(x) = \limsup_{y\to x} \frac{|\phi(x)-\phi(y)|}{d(x,y)}.$$

In addition, here and in the sequel, $\|\nabla\phi\|_\infty = \sup_{x\in E}|\nabla\phi|(x)$.

Definition (Asymptotic log-Harnack inequality)

We say $(P_t)_{t\geq 0}$ satisfies an asymptotic log-Harnack inequality if there exist two non-negative functions $\Phi(\cdot,\cdot)$ on $E\times E$ and $\Psi(\cdot,\cdot,\cdot)$ on $[0,\infty)\times E\times E$ satisfying $\Psi(\cdot,\cdot,\cdot)\to 0$ as $t\to\infty$ such that

$$P_t\log\phi(y) \leq \log P_t\phi(x) + \Phi(x,y) + \Psi(t,x,y)\|\nabla\log\phi\|_\infty,\ \ t>0 \quad (1.5)$$

holds for any $x,y\in E$ and any positive $\phi\in B_b(E)$ with $\|\nabla\log\phi\|_\infty<\infty$.

Dimension-free Harnack inequalities

- It is first introduced by F.-Y. Wang (1997) to study the log-Sobolev inequality for diffusion processes on Riemannian manifolds.
- Very powerful tool to the study of various important properties of diffusion semigroups.
  Strong Feller property, asymptotic strong Feller property, irreducibility, uniqueness of invariant measure.
  Hypercontractivity, ultracontractivity, estimates on the heat kernels, Varadhan type small time asymptotics, entropy-cost inequality.
- Harnack inequality with power: Kawabi (2005), Wang (2007), Liu (2009), Es-Sarhir, von Renesse, Scheutzow (2009), Da Prato, Röckner and Wang (2009), Wang (2013), Xie (2019), $\cdots$
- The log-HI: Röckner and Wang (2010), Wang-Wu-Xu (2011) (Stoc. Burgers equation), Wang-Zhang (2014), Xie (2018).
- Asy log-HI : implies asymptotical strong Feller property, Xu (2011), Li-Liu-Y.C. Xie (2019), Bao-Wang-Yuan (2019).

For a pseudo-metric d_p on E and probability measure μ_1, μ_2 on E, let$\|\mu_1 - \mu_2\|_{d_p} = \inf_{\mu \in \mathcal{C}(\mu_1,\mu_2)} \int_{E\times E} d_p(x,y)\mu(dx,dy)$, where $\mathcal{C}(\mu_1,\mu_2)$ denotes the collection of probability measures on $E \times E$ with marginals μ_1 and μ_2. Let $\{d_n\}_{n=1}^{\infty}$ be a totally separating systems of pseudo-metrics for E, that is, for any $m < n$ and $x, y \in E$, $d_m(x,y) \leq d_n(x,y)$, and for any $x \neq y$ $\lim_{n\to\infty} d_n(x,y) = 1$.

Definition (Definition 3.1, Hairer and Mattingly (2006))

$(P_t)_{t\geq 0}$ on (E,d) is said to be asymptotically strong Feller at point $x \in E$ if there exists a totally separating systems of pseudo-metrics $\{d_n\}_{n=1}^{\infty}$ on E and a positive sequence $\{t_n\}_{n=1}^{\infty}$ such that

$$\inf_{B\in\mathcal{B}_x} \limsup_{n\to\infty} \sup_{y\in B} \|P_{t_n}1_B(x) - P_{t_n}1_B(y)\|_{d_n} = 0.$$

Furthermore, if this property holds for any $x \in E$, then $(P_t)_{t\geq 0}$ is called asymptotically strong Feller.

Stochastic heat equation with reflection

Definition (Definition of the solution to (1.1))

Let $h \geq 0$. A pair (u, η) is said to be a solution of (1.1) *if*

(i) $(u(t, \cdot))_{t \geq 0}$ *is $\mathscr{F}_t$-adapted and* $u(t, \theta) \geq 0, t \geq 0, \theta \in [0, 1]$ *a.s.*

(ii) η *is a positive random measure on* $[0, \infty) \times [0, 1]$ *such that*

(a) $\eta(\{t\} \times (0, 1)) = 0,\ t \geq 0,$ $\int_0^t \int_0^1 \theta(1-\theta)\eta(dsd\theta) < \infty,\ t \geq 0;$

(b) η *is $\mathscr{F}_t$-adapted.*

(iii) *For any $t \geq 0$ and $\phi \in C_0^2(0, 1)$,*

$$\begin{aligned}\langle u(t), \phi\rangle =& \langle h, \phi\rangle + \frac{1}{2}\int_0^t \langle u(s), \phi''\rangle ds + \int_0^t \langle b(u(s)), \phi\rangle ds \\ &+ \int_0^t \int_0^1 \phi(\theta) W(dsd\theta) + \int_0^t \int_0^1 \phi(\theta)\eta(dsd\theta)\end{aligned}$$

(iv) η *is supported on* $\{(t, \theta) : u(t, \theta) = 0\}$*, that is,*

$\int_0^\infty \int_0^1 u(t, \theta)\eta(dtd\theta) = 0\ a.s.$

Let $K_0 := \{h \in H : h(\theta) \geq 0, \theta \in [0,1]\}$ and let P_t denote the Markov semigroup acting on $B_b(K_0)$ associated with the reflected SPDE (1.1), that is, $P_t\Phi(h) = \mathbb{E}[\Phi(u(t;h))],\ t \geq 0, \Phi \in B_b(K_0)$.

Theorem (Xie [7])

- *For any* $0 \leq \Phi \in B_b(K_0)$, *for all* $h, \tilde{h} \in K_0$ *and* $t > 0$

$$\left(P_t\Phi(\tilde{h})\right)^p \leq P_t\Phi^p(h)\exp\left\{p^*|h-\tilde{h}|^2\left(\frac{L^2(e^{\pi^2 t}+1)+\pi^4}{\pi^2(e^{\pi^2 t}-1)} - \frac{2L^2te^{\pi^2 t}}{(e^{\pi^2 t}-1)^2}\right)\right\}$$

- *The Markov semigroup* $(P_t)_{t\geq 0}$ *has a unique invariant probability measure* μ *such that* $\mu\left(\exp\left(\delta|\cdot|^2\right)\right) < \infty$ *holds for for some* $\delta > 0$.
- P_t *is hypercontractive with respect to* μ.

Theorem (Xie [7])

In particular, for large enough t, P_t is compact in $L^2(\mu)$ and $\|P_t\|_{L^p(\mu)\to L^q(\mu)} < \infty$, which implies that (P_t) is hyperbounded.

Definition

Markov semigroup $(P_t)_{t\geq 0}$ is said to be hypercontractive if it has an invariant probability measure μ and for some $t > 0$

$$\|P_t\|_{L^2(\mu)\to L^4(\mu)} := \sup\{\|P_t\Phi\|_{L^4(\mu)} : \ \mu(\Phi^2) \leq 1\} = 1.$$

If $\|P_t\|_{L^2(\mu)\to L^4(\mu)} < \infty$ for some $t > 0$, then (P_t) is hyperbounded.

The equival. of the hypercontra. and the LSI is known (L. Gross (1975)).

Theorem (Xie [7])

(i) *(**Exponential convergence of entropy**)*
There exist two constants $C, \lambda > 0$ *such that for all* $t \geq 0$ *and positive* $\Phi \in B_b(K_0)$ *with* $\mu(\Phi) = 1$

$$\mu((P_t\Phi)\log(P_t\Phi)) \leq C\exp(-\lambda t)\mu(\Phi\log\Phi).$$

(ii) *(**Exponential convergence to** μ *in* $L^2(\mu)$)*
There exists a constant $C > 0$ *such that for all* $t \geq 0$ *and* $\Phi \in L^2(\mu)$

$$\|P_t\Phi - \mu(\Phi)\|_{L^2(\mu)} \leq C\exp(-\kappa t)\|\Phi - \mu(\Phi)\|_{L^2(\mu)}.$$

(iii) *(**Exponential convergence in total variation norm**)*
There exist two constants $T, C > 0$ *such that for all* $t \geq T$

$$\|\mu_t^h - \mu_t^{\tilde{h}}\|_{\mathrm{TV}} \leq C\exp(-\kappa t)|h - \tilde{h}|,\ \ h, \tilde{h} \in K_0.$$

Stochastic Cahn-Hilliard equation

Example

(1) (Example 3.14 (Hairer and Mattingly 2006))

$$dX(t) = (X(t) - X^3(t))dt + dB(t), \qquad dY(t) = -Y(t)dt.$$

(2) (Example 3.15 (Hairer and Mattingly 2006)) Consider the Ornstein-Uhlenbeck process $u(t,x) = \sum \hat{u}(t,k)exp(ikx)$ with

$$d\hat{u}(t,k) = -(1 + |k|^2)\hat{u}(t,k)dt + \exp(-|k|^3)dB^c(t).$$

The Markov semigroups above are not strong Feller, but are asymptotic strong Feller.
Strong Feller property depends on the random effects.
Asymptotic strong Feller depends on the contraction of dynamics.
Two kinds of noises will be considered in the following.

Notation

- $Au = \frac{\partial^2 u}{\partial \theta^2}$ with $D(A) := \{u \in H^2(0,1) : \ u'(0) = u'(1) = 0\}$.
- $(-A)^{\frac{\gamma}{2}} u = \sum_{n=1}^{\infty} (n\pi)^{2\gamma} u_n e_n$ for any $u = \sum_{n=0}^{\infty} u_n e_n$,
  $V_\gamma = D\left((-A)^{\frac{\gamma}{2}}\right) := \left\{u = \sum_{n=0}^{\infty} u_n e_n : \ \sum_{n=0}^{\infty} (n\pi)^{2\gamma} u_n^2 < \infty\right\}$
- $\mathbf{H} = V_{-1}$ and the affine space $\mathbf{H}^c = \{u \in \mathbf{H} : \ \bar{u} = c\}$.

Definition (Definition of the solution of (1.2))

(1) *The quadruplet* (u, η_+, η_-, W) *defined on a filtered complete probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}; \mathbb{P})$ *is said to be a weak solution of* (1.2) *if*
(i) $u \in C((0,\infty) \times [0,1]; [-1,1]) \cap C([0,1]; \mathbf{H})$ $a.s.$ *and* $f(u) \in L^1([0,T] \times [0,1])$ $a.s.$ *for any* $T > 0$.
(ii) η_+ *and* η_- *are two positive random measure on* $[0,\infty) \times [0,1]$ *satisfying* $\eta_\pm([\delta, T] \times [0,1]) < \infty$ $a.s.$ *for all* $\delta \in (0,T]$ *and* $T > 0$.
(iii) $(W(t))_{t\geq 0}$ *is a cylindrical Wiener process on* $L^2(0,1)$ *and the stochastic process* $(u(t), W(t))_{t\geq 0}$ *is* $(\mathcal{F}_t)$*-adapted.*

Definition

(iv) *For all* $\phi \in D(A^2)$ *and* $0 < \delta \leq t$,

$$\begin{aligned} &\langle u(t), \phi \rangle \\ =&\langle u(\delta), \phi \rangle - \frac{1}{2} \int_\delta^t \langle u(s), A^2 \phi \rangle ds - \frac{1}{2} \int_\delta^t \langle f(u(s)), A\phi \rangle ds \\ &- \frac{1}{2} \int_\delta^t \int_0^1 A\phi(\theta) \left(\eta_+(dsd\theta) - \eta_-(dsd\theta) \right) - \int_\delta^t \langle \phi', dW(s) \rangle \ a.s. \end{aligned} \tag{3.1}$$

(v) *The contact properties*
$\mathrm{supp}(\eta_+) \subset \{(t, x) \in [0, \infty) \times [0, 1] : \ u(t, \theta) = +1\}$ *and*
$\mathrm{supp}(\eta_-) \subset \{(t, x) \in [0, \infty) \times [0, 1] : \ u(t, \theta) = -1\}$ *a.s.,*

$$\int_0^\infty \int_0^1 (1 + u(t, \theta)) \eta_-(dtd\theta) = \int_0^\infty \int_0^1 (1 - u(t, \theta)) \eta_+(dtd\theta) = 0 \ a.s.$$

(2) (u, η_+, η_-, W) *is said to be a strong one if the stochastic process* $(u(t))_{t \geq 0}$ *is adapted to the filtration* $(\mathcal{F}_t^W)_{t \geq 0}$.

Theorem (Debussche and Goudenège (2011))

For any $c \in (-1,1)$ *and* $x \in K := \{x \in L^2(0,1) : x \in [-1,1]\}$ *with* $\bar{x} = c$, *the SPDE* (1.2) *has a unique strong solution* $(u(t;x);\eta_+,\eta_-,W)$.
(i) *The average of* $u(t;x)$ *is conservative in* t, *that is,* $\bar{u}(t;x) = \bar{x}$.
(ii) $(u(t;x); t \geq 0, x \in K \cap \mathbf{H}^c)$ *is a* $K \cap \mathbf{H}^c$*-valued continuous Markov process and its associated Markov transition semigroup* P_t^c *is strong Feller.*
(iii) *For each* $c \in (-1,1)$,

$$\nu^c(dx) = \frac{1}{Z^c} \exp\left(-\int_0^1 F(x(\theta))d\theta\right) 1_K(x)\mu_c(dx)$$

is the unique invariant measure of P_t^c, *where* μ^c *denotes the Gaussian measure* $N(ce_0,(-A)^{-1})$ *and* Z^c *denotes the normalization constant.*
(iv) *For any* $k \in \mathbb{N}$ *and* $0 = t_0 < t_1 < t_2 < \cdots < t_k$, $(u^n(t_i;x))_{i=1}^k$ *converges weakly to* $(u(t_i;x))_{i=1}^k$. *For any* $\phi \in B_b(\mathbf{H}^c)$ *and* $t \geq 0$, *we have* $\lim_{n\to\infty} P_t^{n,c}\phi(x) = P_t^c\phi(x)$.

Theorem (Goudenège, Xie (2019))

Suppose $\pi^2 > \lambda$. Then the Harnack inequality with power $p > 1$

$$|P_t^c \phi|^p(y) \leq \frac{1}{2} P_t^c |\phi|^p(x) \exp\left\{ \frac{p(\pi^2 - \lambda)\pi^2 |x - y|_{-1}^2}{(p-1)(e^{(\pi^2-\lambda)\pi^2 t} - 1)} \right\}$$

holds for any $\phi \in B_b(\mathbf{H}^c)$, $x, y \in K \cap \mathbf{H}^c$ and $t > 0$. In particular, the log-Harnack inequality

$$P_t^c \log \phi(y) \leq \frac{(\pi^2 - \lambda)\pi^2 |x - y|_{-1}^2}{2(e^{(\pi^2-\lambda)\pi^2 t} - 1)} + \log P_t^c \phi(x)$$

holds for any $0 < \phi \in B_b(\mathbf{H}^c)$, $x, y \in K \cap \mathbf{H}^c$ and $t > 0$.

Remark

- *Let* $B=(-A)^{\frac{1}{2}}$*. Then* B *is reversible on* $\mathrm{span}\{e_i : i=1,2,\cdots\}$ *and*

$$|B^{-1}z|^2=|z|_{-1}^2,\ z\in\mathbf{H}^0.$$

- *If we consider* $B=\frac{d}{d\theta}$ *with* $\mathrm{Dom}(B)=H^1(0,1)$*, then we can show the following equation*

$$|B^*(BB^*)^{-1}z|^2=|z|_{-1}^2,\ z\in\mathbf{H}^0.$$

- *The method can be applied to the SPDE* (1.2) *with more general* B *instead of* $B=\frac{d}{d\theta}$ *or* $B=(-A)^{\frac{1}{2}}$*. In fact, if* BB^* *is reversible restricted on* $\mathrm{span}\{e_n : n=1,2,\cdots\}$ *with* $|B^*BB^*z|\le C|z|_{-1}, z\in\mathbf{H}$ *for some* $C>0$ *and* (1.2) *has a unique solution, then the Harnarck equalities can be established. For example, if there exists a strictly positive sequence* $\{b_n\}_{n=1}^\infty$ *such that* $Be_n=b_ne_n, n=1,2,\cdots$ *and the sequence* $\{nb_n^{-1}\}_{n=1}^\infty$ *is bounded, then* B *satisfies the assumptions stated above.*

The case of highly degenerate noise

Assumption H

- B is a Hilbert-Schmidt operator from $L^2(0,1)$ to $\mathbf{H}$, which it is equivalent to the fact that $B(-A)^{-1}B^*$ is a trace class on $L^2(0,1)$.
- $B^*e_0 = 0$.
- There exists a non-negative sequence $\{b_i\}_{i=1}^{\infty}$ such that $Bu = \sum_{i=1}^{\infty} b_i \langle u, e_i \rangle$ and there exists a big enough integer N such that $b_i > 0,\ i = 1, 2, \cdots, N$ and $(N+1)^2\pi^2 > \lambda$.
  It is known as the essentially elliptic condition introduced by Hairer and Mattingly (2006).

Let $p_n(u) = 2\sum_{i=0}^{n} \frac{u^{2i+1}}{2i+1},\ u \in \mathbb{R}$. Then it is a non-decreasing $(2n+1)$-degree polynomial and $-p_n(u) + \lambda u$ converges to $f(u)$ for $u \in (-1,1)$.

Let $\{u^n\}$ be the solution of the SPDE

$$\begin{cases} \dfrac{\partial u^n}{\partial t}(t,\theta) = -\dfrac{1}{2}\dfrac{\partial^2}{\partial \theta^2}\left(\dfrac{\partial^2 u^n}{\partial \theta^2}(t,\theta) - p_n(u^n(t,\theta)) + \lambda u^n(t,\theta)\right) \\ \qquad\qquad + B\dot{W}(t,\theta), \quad t>0,\ \theta\in(0,1), \\ \dfrac{\partial u^n}{\partial \theta}(t,0) = \dfrac{\partial u^n}{\partial \theta}(t,1) = \dfrac{\partial^3 u^n}{\partial \theta^3}(t,0) = \dfrac{\partial^3 u^n}{\partial \theta^3}(t,1) = 0,\ t\geq 0, \\ u^n(0,\theta) = x(\theta),\ \theta\in(0,1). \end{cases} \tag{3.2}$$

Theorem (Goudenège, Manca (2015))

For any $c\in(-1,1)$, the following hold:
(i) *There exists a subsequence n_k and a Markov semigroup $(P_t^c)_{t\geq 0}$ such that $\lim_{k\to\infty} P_t^{c,n_k}\phi(x) = P_t^c\phi(x)$ holds for any $x\in\mathbf{H}^c$ and any $\phi\in B_b(\mathbf{H}^c)$.*
(ii) *$(P_t^c)_{t\geq 0}$ has an invariant probability measure $\tilde{\mu}^c$.*

Remark

- *We will fix a converging subsequence* $P_t^{n_k,c}$*. For simplicity, we will still use* $P_t^{n,c}$ *and* $u^n(t)$ *instead of* $P_t^{n_k,c}$ *and* $u^{n_k}(t)$*.*
- *Let* $u(t;x)$ *denote the Markov process associated with* $(P_t^c)_{t\geq 0}$*.*
- *Formally speaking, the sequence* $\{u^n\}_{n=1}^{\infty}$ *converges to the solution of* (1.2)*. But any limit of* $\{u^n\}_{n=1}^{\infty}$ *can not be characterized as a solution of SPDEs.*

Theorem (Goudenège, Xie (2019))

Let $c \in (-1,1)$*. For any* $\varsigma > 0$ *satisfying* $\pi^4 > 2\varsigma\|B^*\|^2$*, the invariant measure* $\tilde{\mu}^c$ *satisfies the exponential integrability* $\tilde{\mu}^c\left(\exp(\varsigma|\cdot|_{-1}^2)\right) < \infty$*, where* $\|B^*\|$ *denotes the operator norm of* B^**.*
If further $\pi^4 > \lambda$*, then* $\tilde{\mu}^c$ *is the unique invariant measure and for any Lipschitz continuous function* $\phi \in B_b(\mathbf{H}^c)$*,*

$$|P_t^c\phi(x) - \tilde{\mu}^c(\phi)| \leq \|\nabla\phi\|_\infty \exp^{-(\pi^4-\lambda)t}\left(|x|_{-1} + \tilde{\mu}(|\cdot|_{-1})\right),\ x \in \mathbf{H}^c.$$

Since we are considering the degenerate noise, the strong Feller can not be expected. So we study the asymptotic log-HL.

Theorem (Goudenège, Xie (2019))

Suppose Assumption H is satisfied. Then, for any $c \in (-1,1)$*, the Markov semigroup* $(P_t^c)_{t\geq 0}$ *satisfies the asymptotic log-Harnack inequality. More precisely, we have that*

$$P_t^c \log \phi(y) \leq \log P_t^c \phi(x) + \frac{\lambda}{8\alpha}(1-\exp(-2\alpha t))\|B^{-1}A\Pi_l\|_{op}^2 |x-y|_{-1}^2 + \exp(-\alpha t)\|\nabla \log \phi\|_\infty |x-y|_{-1}, \; t>0$$

holds for any $x, y \in \mathbf{H}^c$ *and any positive* $\phi \in B_b(\mathbf{H}^c)$ *with* $\|\nabla \log \phi\|_\infty < \infty$*, where* $\|B^{-1}A\Pi_l\|_{op}$ *denotes the operator norm of* $B^{-1}A\Pi_l$ *from* $\mathbf{H}^c$ *to the* N*-dimensional space* $\mathrm{span}\{e_1, e_2, \cdots, e_N\}$ *and*

$$\alpha = \frac{1}{2}\min\left\{\pi^4, \left((N+1)^2\pi^2 - \lambda\right)(N+1)^2\pi^2\right\}.$$

Corollary

(i) $(P_t^c)_{t\geq 0}$ *is asymptotically strong Feller.*
(ii) *For any Lipschitz continuous function* $\phi \in B_b(\mathbf{H}^c)$*,*

$$|\nabla P_t^c \phi| \leq \left(\frac{\lambda}{4\alpha}\right)^{\frac{1}{2}} \|B^{-1}A\Pi_l\|_{op} \sqrt{P_t^c \phi^2 - (P_t^c \phi)^2} + \|\nabla \phi\|_\infty \exp(-\alpha t).$$

(iii) *For any non-negative* $\phi \in B_b(\mathbf{H}^c)$ *with* $\|\phi\|_\infty < \infty$ *and all* $x \in \mathbf{H}^c$*,*

$$\limsup_{t\to\infty} P_t^c \phi(x) \leq \log\left(\frac{\tilde{\mu}^c(\exp \phi)}{\int_{\mathbf{H}^c} \exp(-\frac{\lambda}{8\alpha}\|B^{-1}A\Pi_l\|_{op}^2 |x-y|_{-1}^2)\tilde{\mu}^c(dy)}\right),$$

where $\tilde{\mu}^c$ *the invariant measure of* P_t^c*.*
(iv) *If for some* $x \in \mathbf{H}^c$ *and* $A \subset \mathbf{H}^c$*,* $\liminf_{t\to\infty} P_t^c(x, A) > 0$ *holds, then, for any* $y \in \mathbf{H}^c$ *and* $\epsilon > 0$*,* $\liminf_{t\to\infty} P_t^c(y, A_\epsilon) > 0$*.*

Remark

(i) *From the asymptotical strong Feller property, it follows that any two different ergodic invariant measures must have disjoint topological supports, see Theorem 3.16 (Hairer and Mattingly (2006)).*
Moreover, combining with weakly topological irreducibility, we have the uniqueness of invariant measure.
(ii) *We can also show that for any Lipschitz continuous function* $\phi \in B_b(\mathbf{H}^c)$,

$$|\nabla P_t^c \phi|(x) \leq \left(\frac{\lambda}{4\alpha}\right)^{\frac{1}{2}} \|B^{-1}A\Pi_l\|_{op}\|\phi\|_\infty + 2\|\nabla\phi\|_\infty \exp(-\alpha t),$$

which is a sufficient condition for the asymptotical strong Feller property (Hairer and Mattingly (2006)).

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Thank you for your kind attention!