Asymptotic behavior of reflected SPDEs with singular potentials driven by additive noises

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In this talk, we will mainly review the results on the asymptotic behavior of the following reflected SPDE with a singular potential by making use of the dimension-free Harnack inequality:

$$\frac{\partial u}{\partial t}(t,\theta) = -\frac{1}{2}(-\Delta)^{\gamma}\left(\frac{\delta \mathcal{H}(u)}{\delta u(\theta)}(t,\theta) + \xi(t,\theta)\right) + (-\Delta)^{\gamma/2} B\dot{W}(t,\theta). \tag{1}$$

Hereafter, $\gamma \in \{0,1\}$, $\theta \in (0,1)$, $\xi(t,\theta)$ denotes a signed random reflecting measure which prevents the solution from leaving a subset $\mathcal{I}$ of $\mathbb{R}$, $\dot{W}(t,\theta)$ denotes a Gaussian noise and $\frac{\delta \mathcal{H}(u)}{\delta u(\theta)}$ denotes the functional derivative of the formal Hamiltonian $\mathcal{H}$, where

$$\mathcal{H}(u) = \int_0^1 \left(\frac{1}{2}|\nabla u(\theta)|^2 + F(u(\theta))\right) d\theta$$

is called the Ginzburg-Landau-Wilson free energy with a self-potential F.

The SPDE (1) is sometimes called the time-dependent Ginzburg-Landau equation. In addition, because the reflecting term $\xi(t,\theta)$ is considered, (1) is also called the random parabolic obstacle problem and is regarded as the infinite-dimensional Skorokhod problem. Formally, we can easily see that the mass of solutions of (1) (i.e. $\int_0^1 u(t,\theta)d\theta$)) is conservative under the time evolution for the case $\gamma = 1$, whereas it is not conservative for the case $\gamma = 0$. Hence, according to Hohenberg and Halperin (1977), we will call (1) with $\gamma = 0$ the Model A with reflection, and respectively (1) with $\gamma = 1$ the Model B with reflection in the following. As for applications, the reflected SPDE (1) has been used to model the fluctuations for $\nabla\phi$ interface models on a hard wall with or without conservation of the area. In this talk, both the Model A and the Model B will be discussed.

For the Model A with reflection, we will study (1) with $\mathcal{I} = [0,\infty)$ and $B = I$ under Dirichlet boundary conditions. Noting that $\frac{\delta \mathcal{H}(u)}{\delta u(\theta)} = -\Delta u(\theta) + F'(u(\theta))$, we know that, under the above assumptions, (1) can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\theta) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2}(t,\theta) - F'(u(t,\theta)) + \dot{W}(t,\theta) + \xi(dtd\theta),\ \theta \in (0,1), \\ u(t,0) = u(t,1) = 0, \\ u(0,\theta) = h(\theta) \geq 0,\ \theta \in (0,1), \\ u(t,\theta) \geq 0,\ \theta \in [0,1]\ a.s, \end{cases} \tag{2}$$

which is called the reflected stochastic reaction-diffusion equation. Under very weak conditions, we study the hypercontractive property of the Markov semigroup associated with (2) driven by the additive space-time white noise. To show it, the coupling of property, the Harnack inequality with power of the Markov semigroup and the Gaussian concentration of the invariant probability measure are investigated respectively. As the same time, we can show the compactness of the Markov semigroup, the exponential convergences of the Markov semigroup to its unique invariant measure in the sense of L^2, total variation norm and entropy are obtained.

In order to explain the Model B with reflection, let us first define the non-linear function f of the logarithmic type by

$$f(u) = \log\left(\frac{1-u}{1+u}\right) + \lambda u,\ u \in (-1,1),$$

which has two singularity points -1 and 1 and let $F'(u) = -f(u)$. Then (1) with $\mathcal{I} = [-1,1]$ is called the stochastic Cahn-Hilliard equation with logarithmic potential and particularly under Neumann boundary conditions, it can be written as the following:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\theta) =& -\dfrac{1}{2}\dfrac{\partial^2}{\partial\theta^2}\left(\dfrac{\partial^2 u}{\partial\theta^2}(t,\theta) + f(u(t,\theta)) + \xi(t,\theta)\right) + B\dot{W}(t,\theta), \\ u(t,0) =& u(t,1) = \dfrac{\partial^3 u}{\partial x^3}(t,0) = \dfrac{\partial^3 u}{\partial x^3}(t,1) = 0,\ t \geq 0, \\ u(0,\theta) =& x(\theta),\ \theta \in (0,1), \\ u(t,\theta) \in& [-1,1]\ a.s., \end{cases} \tag{3}$$

where $\xi(t,\theta) = \eta_-(t,\theta) - \eta_+(t,\theta)$ preventing the solution $u(t,\theta)$ from exiting $[-1,1]$. In fact, in applications, the solution of (3) is explained as the rescaled density of atoms or concentration of one of material's components which naturally takes values in $[-1,1]$. Hence, the function f defined above is more important and owing to the effect of noises, reflecting measures $\eta_-(t,\theta)$ and $\eta_+(t,\theta)$ are required. We will study the asymptotic behavior of the solution of (3) driven by both the degenerate colored noise and the non-degenerate white noise. For the case of degenerate colored noise, the asymptotic log-Harnack inequality is established under the so-called essentially elliptic conditions, which implies the asymptotic strong Feller property. For the case of non-degenerate space-time white noise, the Harnack inequality with power is established. This part is the joint work with L. Goudenège.