Asymptotic behavior of the spectral functions for Schrödinger forms

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1 Setting and the main result

Let $\{X_t\}$ be the rotationally invariant α-stable process on $\mathbb{R}^d$ with $0 < \alpha < 2$ and denote by $(\mathcal{E}, \mathcal{F})$ the corresponding Dirichlet form on $L^2(\mathbb{R}^d)$. We assume $\alpha < d$, transience of $\{X_t\}$ and denote the Green kernel by $G(x, y)$. Let μ and ν be positive Radon smooth measures satisfying three properties, i.e. Kato class, Green tightness and of finite 0-order energy integral. Define the Schrödinger form by

$$\mathcal{E}^\lambda(u, v) = \mathcal{E}(u, v) - \int_{\mathbb{R}^d} u(x)v(x)\mu(dx) - \lambda \int_{\mathbb{R}^d} u(x)v(x)\nu(dx) \qquad (\lambda \geq 0)$$

For simplicity, we also assume μ is critical, that is,

$$\inf \left\{ \mathcal{E}(u, u) \;\middle|\; u \in \mathcal{F}_e, \quad \int_{\mathbb{R}^d} u^2(x)\mu(dx) = 1 \right\} = 1.$$

Here $\mathcal{F}_e$ is the extended Dirichlet space. Define the spectral function by

$$C(\lambda) = -\left\{ \mathcal{E}^\lambda(u, u) \;\middle|\; \int_{\mathbb{R}^d} u^2(x)dx = 1 \right\}.$$

There are several preceding results for the differentiability of the spectral functions. Takeda and Tsuchida [2] treated this problem in the framework of $\mu = \nu$. Nishimori [1] treated the differentiability of $C(\lambda)$. Both of them showed that the differentiability of the spectral function is equivalent to $d/\alpha \leq 2$. In this talk, we treat the precise asymptotic behavior of the spectral function and our main result is as follows:

Theorem 1. *(W. 2018)*
As $\lambda \downarrow 0$, the spectral function $C(\lambda)$ satisfies the asymptotic behavior as follows:

$$C(\lambda) \sim \left(\frac{\alpha\Gamma(\frac{d}{2})|\sin(\frac{d}{\alpha}\pi)|\langle h_0, h_0\rangle_\nu}{2^{1-d}\pi^{1-\frac{d}{2}}\langle \mu, h_0\rangle^2}\lambda \right)^{\frac{\alpha}{d-\alpha}} \qquad (1 < d/\alpha < 2)$$

$$C(\lambda) \sim \frac{\Gamma(\alpha + 1)\langle h_0, h_0\rangle_\nu}{2^{1-d}\pi^{-\frac{d}{2}}\langle \mu, h_0\rangle^2} \cdot \frac{\lambda}{\log \lambda^{-1}} \qquad (d/\alpha = 2)$$

$$C(\lambda) \sim \frac{\langle h_0, h_0 \rangle_\nu}{\langle h_0, h_0 \rangle_m} \cdot \lambda \quad (d/\alpha > 2)$$

Here $h_0(x)$ is the ground state of $\mathcal{E}^0$ and m stands for the Lebesgue measure of $\mathbb{R}^d$.

Remark 2. For $\mu = \nu = V \cdot m$, this result is the same as in [3]

2 Outline of the proof

(1) Let $G_\beta(x, y)$ be the resolvent kernel of $\{X_t\}_{t \geq 0}$. Define the compact operators by

$$K_\lambda f(x) = \int_{\mathbb{R}^d} G_{C(\lambda)}(x, y) f(y)(\mu + \lambda\nu)(dy) \quad f \in L^2(\mu + \lambda\nu)$$
$$\tilde{K}_\lambda f(x) = \int_{\mathbb{R}^d} G_{C(\lambda)}(x, y) f(y)\mu(dy) \quad f \in L^2(\mu)$$

(2) Denote the principal eigenfunction of these operator by h_λ and $\tilde{h}_\lambda$. The principal eigenvalue of K_λ is 1, while the principal eigenvalue of $\tilde{K}_\lambda$ admits the asymptotic behavior as follows.

$$\lim_{\lambda \to 0} \frac{1 - \gamma_{C(\lambda)}}{k(C(\lambda))} = \kappa(d, \alpha, \mu) \qquad k(\beta) = \begin{cases} \beta^{d/\alpha - 1} & (1 < d/\alpha < 2) \\ \beta \log \beta^{-1} & (d/\alpha = 2) \\ \beta & (d/\alpha > 2) \end{cases}$$

Here $\kappa(d, \alpha, \mu)$ is a unique positive constant.

(3) Considering the inner product of h_λ and $\tilde{h}_\lambda$, we have

$$(1 - \gamma_{C(\lambda)})\langle h_\lambda, \tilde{h}_\lambda \rangle_\mu = \lambda \langle h_\lambda, \tilde{h}_\lambda \rangle_\nu.$$

Both h_λ and $\tilde{h}_\lambda$ converges to the ground state h_0 in $L^2(\mu)$ and $L^2(\nu)$. Thus we obtain the desired result.

References

[1] Nishimori, Y.: Large deviations for symmetric stable processes with Feynman-Kac functionals and its application to pinned polymers, Tohoku Math. Journal 65, 467–494, (2013).

[2] Takeda, M. and Tsuchida, K.: Differentiability of spectral functions for symmetric α-stable processes, Trans. Amer. Math. 359, 4031–4054, (2007).

[3] Wada, M.: Asymptotic expansion of resolvent kernels and behavior of spectral functions for symmetric stable processes, J. Math. Soc. Jpn. 69, 673–692, (2017).