Green-tight measure of Kato class and compact embedding theorem for symmetric Markov processes (joint work with Kazuhiro Kuwae)

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- In this talk, we would like to discuss compact embeddings for symmetric Markov processes.
- Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(E; m)$ associated with a m-symmetric Markov process X.
- For a suitable measure μ, Stollmann-Voigt proved the following inequality: for $\alpha > 0$

$$\int u^2 d\mu \leq \|R_\alpha \mu\|_\infty \mathcal{E}_\alpha(u, u), \quad u \in \mathcal{F}, \tag{1}$$

  where R_α is the α-resolvent of X and $\mathcal{E}_\alpha(u, u) = \mathcal{E}(u, u) + \alpha(u, u)$. Moreover, if X is transient, (1) holds for $\alpha = 0$ and $u \in \mathcal{F}_e$.
- Hence the embedding $(\mathcal{F}, \mathcal{E}_1) \hookrightarrow L^2(\mu)$ (or $(\mathcal{F}_e, \mathcal{E}) \hookrightarrow L^2(\mu)$ if X is transient) is continuous.

- Takeda introduced following conditions:

(I) X is irreducible:
If any Borel set B satisfies $P_t 1_B u = 1_B P_t u$ for all $u \in L^2(E;m)$ and $t > 0$, then $m(B) = 0$ or $m(B^c) = 0$ holds.

(RSF) X has the resolvent strong Feller property:
$R_\alpha(\mathcal{B}_b) \subset C_b$ for any $\alpha > 0$.

(Tightness) X has a tightness property:
For any $\varepsilon > 0$, there exists a compact set $K(\subset E)$ such that

$$\|R_1(1_{K^c} m)\|_\infty < \varepsilon.$$

If X satisfies conditions (I), (RSF) and (Tightness), X is called "class (T)".

Theorem 1 (Takeda ('19))

Suppose that X is class (T).

(1) **The Markov semigroup is compact on $L^2(E;m)$ and its every eigenfunction has a bounded continuous version.**

(2) **The embedding $(\mathcal{F},\mathcal{E}_1) \hookrightarrow L^2(E;m)$ is compact.**

(3) **If X is transient and $\mu \in S^1_{CK_\infty}(\mathrm{X})$, then the embedding $(\mathcal{F}_e,\mathcal{E}) \hookrightarrow L^2(\mu)$ is compact.**

(4) **There exists a bounded ground state uniquely up to sign, that is, the function ϕ_0 which attains the infimum:**

$$\inf\left\{\mathcal{E}(u,u) : u \in \mathcal{F},\ \int_E u^2 dm = 1\right\}.$$

Moreover, ϕ_0 can be taken to be strictly positive.

Remark 2

(1) **In Theorem 1, (1) $\iff$ (2).**

(2) **If μ is a smooth measure, there exists a PCAF A_t^{μ} under Revuz correspondence.**
For example, if $\mu(dx) = V(x)m(dx)$, its associated additive functional A_t^{μ} is $\int_0^t V(X_s)ds$.
The statement (3) plays very important role to prove the large deviation for additive functionals.

(3) **For (3), Chen-T. proved by another method that this embedding is compact under the existence of Green function and gave examples from wide class of jump-type symmetric Markov processes including relativistic or truncated stable processes. (to appear)**

- **In proofs of compactness, we notice that Takeda does not use (RSF) essentially.**
- **He use (RSF) in proving that m belongs to the class of Green-tight Kato measure in the sense of Chen (in notation $S^1_{CK_\infty}(\mathbf{X}^{(1)})$).**
  ($\mathbf{X}^{(1)}$ means the 1-subprocess of $\mathbf{X}$).
- **We would like to clarify where these conditions are used, and generalize these results.**

E : a locally compact separable metric space
m : a positive Radon measure on E with full support.
$\mathrm{X} = (\mathbb{P}_x, X_t)$: m-symmetric special standard process on E.
$\{P_t, t \geq 0\}$: the semigroup of X.
$(\mathcal{E}, \mathcal{F})$: the quasi-regular Dirichlet form generated by X:

$$\mathcal{F} = \left\{ u \in L^2(m) : \lim_{t \downarrow 0} \frac{1}{t}((I - P_t)u, u)_{L^2(m)} < \infty \right\}$$

$$\mathcal{E}(u, v) = \lim_{t \downarrow 0} \frac{1}{t}((I - P_t)u, v)_{L^2(m)}, \quad u, v \in \mathcal{F}.$$

$(\mathcal{F}_e, \mathcal{E})$: the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$.
R_α : the α-resolvent of X.
$S^1(\mathrm{X})$: the family of positive smooth measures in the strict sense under the absolute continuity condition (AC).

In this talk, we always assume that any measure belongs to $S^1(\mathrm{X})$.

Let $P_t(x, dy)$ be the transition function of X, that is,

$$P_t(x, B) = \mathbb{P}_x(X_t \in B).$$

In the sequel, we use the following notations:

(AC) : for any $t > 0$ and $x \in E$, $P_t(x, dy)$ is absolutely continuous with respect to m.

(SF) : for any $t > 0$, $P_t(\mathcal{B}_b(E)) \subset C_b(E)$.

(RSF) : for any $\alpha > 0$, $R_\alpha(\mathcal{B}_b(E)) \subset C_b(E)$.

It is known that

$$\textbf{(SF)} \Longrightarrow \underbrace{\textbf{(RSF)}}_{\text{Takeda's results}} \Longrightarrow \underbrace{\textbf{(AC)}}_{\text{Our results}}.$$

We define α-potential of ν by

$$R_\alpha \nu(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\alpha t} dA_t^\nu\right], \quad x \in E$$

where A_t^ν is the PCAF associated to $\nu \in S^1(\mathrm{X})$.

Definition 3 (Kato class)

(1) Suppose that X is transient. ν is said to be a Green-bounded ($S_{D_0}(\mathrm{X})$) if $\sup_{x\in E} R\nu(x) < \infty$.

(2) ν is said to be a smooth measure of Kato class $S_K^1(\mathrm{X})$ if

$$\lim_{\alpha\to\infty} \sup_{x\in E} R_\alpha \nu(x) = 0.$$

(3) The local Kato class $S_{LK}^1(\mathrm{X})$ is defined by

$$S_{LK} = \{\nu \in S^1(\mathrm{X}) : 1_K \nu \in S_K^1(\mathrm{X}) \text{ for any } K \text{ cpt.}\}.$$

Definition 4 (Two kinds of Green-tight measure)

Let $\nu \in S^1(\mathrm{X})$ and $\alpha \geq 0$. When $\alpha = 0$, we always assume the transience of X.

(1) **(Zhao) $\nu \in S^1_{K_\infty}(\mathrm{X}) \overset{\text{def.}}{\iff} \nu \in S^1_K(\mathrm{X})$ and for any $\varepsilon > 0$ there exists a compact subset $K = K(\varepsilon)$ of E such that**

$$\sup_{x \in E} R_\alpha(1_{K^c}\nu)(x) < \varepsilon.$$

(2) **(Chen) $\nu \in S^1_{CK_\infty}(\mathrm{X}) \overset{\text{def.}}{\iff}$ for any $\varepsilon > 0$ there exists a Borel subset $K = K(\varepsilon)$ of E with $\nu(K) < \infty$ and a constant $\delta > 0$ such that for all ν-measurable set $B \subset K$ with $\nu(B) < \delta$,**

$$\sup_{x \in E} R_\alpha(1_{B \cup K^c}\nu)(x) < \varepsilon.$$

If $\alpha > 0$, we rewrite $S^1_{K_\infty}(\mathrm{X})$ (resp. $S^1_{CK_\infty}(\mathrm{X})$) with $S^1_{K^+_\infty}(\mathrm{X})$ (resp. $S^1_{CK^+_\infty}(\mathrm{X})$).

Remark 5

(1) **Definition 4(1):** Zhao originally introduced the class $S^1_{K_\infty}(\mathrm{X})$ in considering the gaugeability for d-dim. absorbing Brownian motions $(d \geq 3)$ on bounded open domains.

(2) **Definition 4(2):** However, $S^1_{K_\infty}(\mathrm{X})$ is not enough to develop the gaugeability and subcriticality for symmetric Markov processes. To overcome some difficulty, Chen introduced the class $S^1_{CK_\infty}(\mathrm{X})$.

(3) The Borel set $K = K(\varepsilon)$ in Definition 4(2) can be taken to be a compact set by the inner regularity of m. Hence $S_{CK^{(+)}_\infty}(\mathrm{X}) \subset S_{K^{(+)}_\infty}(\mathrm{X})$.

Remark (continued)

(4) Chen proved that $S^1_{K_\infty^{(+)}}(\mathrm{X}) = S^1_{CK_\infty^{(+)}}(\mathrm{X})$ under (SF). Later, Kim and Kuwae proved the coincidence under (RSF). Moreover, the equality holds under the ultracontractivity of X.

(5) If $\alpha > 0$, $S^1_{K_\infty^+}(\mathrm{X})$ and $S^1_{CK_\infty^+}(\mathrm{X})$ are independent of the choice of $\alpha > 0$ by the resolvent equation.

(6) Chen proved that $(S^1_{CK_\infty}(\mathrm{X}) \subset)S^1_{CK_\infty^+}(\mathrm{X}) \subset S^1_K(\mathrm{X})$.

(7) Clearly, $S^1_{CK_\infty^+}(\mathrm{X}) = S^1_{CK_\infty}(\mathrm{X}^{(1)})$.

- In the sequel, we only consider 0-order Green-tight measure by Remark 5(7).

Theorem 6

Suppose that X satisfies (AC) and $m \in S^1_{CK_\infty}(\mathrm{X}^{(1)})$. Then the L^2-semigroup P_t is a compact operator on $L^2(E;m)$ and its every eigenfunction has a finely continuous Borel measurable bounded m-version. Moreover, if X satisfies (RSF), then every eigenfunction has a bounded continuous m-version.

Theorem 7

Suppose that X satisfies (AC) and $m \in S^1_{CK_\infty}(\mathrm{X}^{(1)})$. Then the embedding $\mathcal{F} \hookrightarrow L^2(E;m)$ is compact.

Theorem 8

Suppose that X is transient and it satisfies (AC). Let $\nu \in S^1_{CK_\infty}(\mathrm{X})$. Then $(\mathcal{F}_e, \mathcal{E})$ is compactly embedded in $L^2(E;\nu)$.

Let λ_2 be the bottom of the spectrum:

$$\lambda_2 := \inf\left\{\mathcal{E}(f,f) : f \in \mathcal{F},\ \int_E f^2 dm = 1\right\}.$$

A function ϕ_0 on E is called a ground state of the L^2-generator for $\mathcal{E}$ if $\phi_0 \in \mathcal{F}$, $\|\phi_0\|_2 = 1$ and $\mathcal{E}(\phi_0, \phi_0) = \lambda_2$.

Theorem 9

Suppose that X satisfies (AC), (I) and $m \in S^1_{CK_\infty}(\mathrm{X})$. Then there exists a bounded ground state ϕ_0 uniquely up to sign. Moreover, ϕ_0 can be taken to be strictly positive on E.

Theorem 10

Suppose that X is transient which possesses (RSF). Take $\nu \in S_{D_0}(\mathrm{X})$ and assume $\nu \notin S_{LK}(\mathrm{X})$.

(1) **If ν has the full quasi-support, then the time changed process $(\check{\mathrm{X}}, \nu)$ does not possess (RSF), but satisfies (AC).**

(2) **There exists a $\beta > 0$ such that the killed process $\mathrm{X}^{-\beta\nu}$ does not possess (RSF), but satisfies (AC).**

Is there a measure ν that satisfies this theorem? $\boxed{\textbf{Yes!}}$

To construct examples, we can apply an example due to Aizenman-Simon (1982).

Example 1 (Brownian motion)

Let X be the d-dimensional BM on $\mathbb{R}^d$ with $d \geq 3$ and m the Lebesgue measure on $\mathbb{R}^d$.
Set $x_n := (2^{-n}, 0, \cdots, 0) \in \mathbb{R}^d$ and $r_n = 8^{-n}$. We set $V_n(x) = 8^{2n} 1_{B_{r_n}(x_n)}(x)$ and $V(x) := \sum_{n=2}^{\infty} V_n(x)$. Then we find that $Vm \in S_{D_0}(\mathrm{X}) \setminus S^1_{LK}(\mathrm{X})$ by Aizenman-Simon ('82). Since X is transient, there exists a function g such that $0 < g \leq 1$ m-a.e. and $Rg \in \mathcal{B}_b(E)$. We put $\nu = (V + g)m$. Then we know that the time-changed processes $\hat{\mathrm{X}}^{\nu}$ associated with ν and the killed process $\mathrm{X}^{-\beta\nu}$ for some $\beta > 0$ do not possess (RSF) by Theorem 10, but satisfy (AC).

Example 2 (stable process)

Take $\alpha \in (0,2)$ and $m \geq 0$. Let $\mathrm{X} = (\Omega, X_t, \mathbb{P}_x)$ be a Lévy process on $\mathbb{R}^d$ with

$$\mathbb{E}_0[e^{i\langle \xi, X_t\rangle}] = \exp\left(-t((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m)\right)$$

If $m > 0$, it is called the relativistic α-stable process with mass m. We assume the transience of X, i.e. $d \geq 3$ with $m > 0$, or $d > \alpha$ with $m = 0$. Let x_n and r_n be the point and constant as in Example 1. We fix $G := B_1(0)$. We set $V_n(x) = 8^{\alpha n} 1_{B_{r_n}(x_n)}(x)$ and $V(x) := \sum_{n=2}^{\infty} V_n(x)$. Then $Vm \in S_{D_0}(\mathrm{X}) \setminus S^1_{LK}(\mathrm{X})$. Hence the killed process $\mathrm{X}^{-\beta\nu}$ for some $\beta > 0$ do not possess (RSF) by Theorem 10, but satisfy (AC).

Example 3 (BM on Riemannian manifold)

**Let (M, g) be a d-dimensional complete smooth Riemannian manifold with $\mathrm{Ric}_g \geq \kappa(d-1)$ for some $\kappa \in \mathbb{R}$.
Let $m = \mathrm{vol}_g$ be the volume measure of (M, g) and Δ_g the Laplace-Bertrami operator of (M, g). Let X be the diffusion process on (M, g) generated by $\frac{1}{2}\Delta_g$. It is known that X is transient if $d \geq 3$. We assume the transience of X. Fix a point $o \in M$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in M such that $2d(x_{n+1}, o) = d(x_n, o)$, $n \in \mathbb{N}$.
We define $V_n(x) := 8^{2n} 1_{B_{r_n}(x_n)}(x)$ and $V(x) := \sum_{n=2}^{\infty} V_n(x)$, where $r_n := 8^{-n}$.
Then we find that $Vm \in S^1_{D_0}(\mathrm{X}) \setminus S_{LK}(\mathrm{X})$. Hence the killed process $\mathrm{X}^{-\beta V m}$ does not possess (RSF) for some $\beta > 0$, but satisfy (AC).**

Thank you

Thank you for your attention !!