Green-tight measures of Kato class and compact embedding theorem for symmetric Markov processes
(joint work with Kazuhiro Kuwae)

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# 1 Introduction

Let $E$ be a locally compact separable metric space and $m$ a Radon measure on $E$ with full support. Let $\mathbf{X}$ be a $m$-symmetric Hunt process and $(\mathcal{E},\mathcal{F})$ its regular Dirichlet form on $L^2(E;m)$. Takeda proved in [3] that the semigroup of $\mathbf{X}$ is compact on $L^2(E;m)$ if $\mathbf{X}$ satisfies that irreducibility **(I)**, resolvent strong Feller property **(RSF)**, and Green-tightness **(T)**. As an application, the compactness of embedding from $(\mathcal{F},\mathcal{E})$ to $L^2(E;m)$ can be proved. This compact embedding theorem plays important role in proving the large deviation principle for additive functionals generated by $\mathbf{X}$. In this talk, we extend this result to the Markov process which satisfies absolute continuity **(AC)** and $m \in S^1_{CK_\infty}(\mathbf{X})$, where $S^1_{CK_\infty}(\mathbf{X})$ is a class of Green-tight measures introduced by Chen ([1]). Finally, we present some examples that are **(AC)** but not **(RSF)**.

# 2 Setting

Let $\mathbf{X} = (\mathbb{P}_x, X_t, \zeta)$ be a $m$-symmetric special standard process on $E$, where $\zeta$ is the life time of $\mathbf{X}$. Let $(\mathcal{E},\mathcal{F})$ be the Dirichlet form associated with $\mathbf{X}$, which is known to be quasi-regular. $\mathbf{X}^{(1)}$ denotes the 1-subprocess of $\mathbf{X}$ defined by $\mathbf{X}^{(1)} = (\mathbb{P}^{(1)}_x, X_t)$ with $\mathbb{P}^{(1)}_x(X_t \in A) = e^{-t}\mathbb{P}_x(X_t \in A)$ for all $t > 0$ and $A \in \mathcal{B}(E)$. Let $(P_t)_{t\geq 0}$ be the transition semigroup of $\mathbf{X}$. The transition kernel of $\mathbf{X}$ is denoted by $P_t(x, dy),\ t > 0$, that is, for any $f \in \mathcal{B}_b(E)$,

$$P_t f(x) := \mathbb{E}_x[f(X_t) : t < \zeta] = \int_E f(y)P_t(x, dy), \quad x \in E,\ t > 0.$$

We assume that $\mathbf{X}$ has the absolute continuity condition, that is, for any Borel set $B$, $m(B) = 0$ implies $P_t(x, B) = \mathbb{P}_x(X_t \in B) = 0$ for all $t > 0$ and $x \in E$. Let $(R_\alpha)_{\alpha>0}$ be the resolvent of $\mathbf{X}$, that is, for any $f \in \mathcal{B}_b(E)$,

$$R_\alpha f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\alpha t} f(X_t)dt\right] = \int_0^\infty e^{-\alpha t} P_t f(x)dt.$$

Here $\mathbf{X}$ is said to possess the *resolvent strong Feller property* **(RSF)** (resp. *strong Feller property* **(SF)**) if $R_\alpha(\mathcal{B}_b(E)) \subset C_b(E)$ for any $\alpha > 0$ (resp. $P_t(\mathcal{B}_b(E)) \subset C_b(E)$ for any $t > 0$). It is known that the implication **(SF)** $\Longrightarrow$ **(RSF)** $\Longrightarrow$ **(AC)** holds. Let $S_1(\mathbf{X})$ be the family of positive smooth measures in the strict sense under **(AC)**. For $\nu \in S_1(\mathbf{X})$, we set

$$R_\alpha \nu(x) := \mathbb{E}_x\left[\int_0^\infty e^{-\alpha t} dA^\nu_t\right],\ x \in E,$$

where $A^\nu_t$ is the positive continuous additive functional associated to $\nu \in S_1(\mathbf{X})$.

**Definition 2.1.** A measure $\nu \in S_1(\mathbf{X})$ is said to be in the *Kato class* if $\lim_{\alpha\to\infty} \sup_{x\in E} R_\alpha \nu(x) = 0$. A measure $\nu \in S_1(\mathbf{X})$ is said to be in the *local Kato class* if for any compact subset $K$ of $E$, $1_K\nu$ is of Kato class.

We denote by $S^1_K(\mathbf{X})$ (resp. $S^1_{LK}(\mathbf{X})$) the family of measures of Kato class (resp. local Kato class).

**Definition 2.2** ([1])**.** Let $\nu \in S_1(\mathbf{X})$ and take an $\alpha \geq 0$. When $\alpha = 0$, we always assume the transience of $\mathbf{X}$. $\nu$ is said to be an *$\alpha$-order Green-tight smooth measure of Kato class with respect to* $\mathbf{X}$ if for any

$\varepsilon > 0$ there exists a Borel subset $K = K(\varepsilon)$ of $E$ with $\nu(K) < \infty$ and a constant $\delta > 0$ such that for all $\nu$-measurable set $B \subset K$ with $\nu(B) < \delta$,

$$\sup_{x \in E} R_\alpha(1_{B \cup K^c}\nu) < \varepsilon.$$

In view of the resolvent equation, for a positive constant $\alpha$, the $\alpha$-order Green-tightness of Kato class is independent of the choice of $\alpha > 0$. Let denote by $S^1_{CK_\infty}(\mathbf{X})$ (resp. $S^1_{CK^+_\infty}(\mathbf{X})$) the family of 0-order (resp. positive order) Green-tight smooth measure of Kato class. Clearly, $S^1_{CK^+_\infty}(\mathbf{X}) = S^1_{CK_\infty}(\mathbf{X}^{(1)})$. It can be proved as in [1] that $S^1_{CK^+_\infty}(\mathbf{X}) \subset S^1_K(\mathbf{X})$ and $S^1_{CK_\infty}(\mathbf{X}) \subset S^1_K(\mathbf{X})$. It is easy to see that $S^1_{CK_\infty}(\mathbf{X}) \subset S^1_{CK_\infty}(\mathbf{X}^{(1)})$ if $\mathbf{X}$ is transient.

# 3 Results

**Theorem 3.1** ([2, Theorem 1.5])**.** *Suppose that* $\mathbf{X}$ *satisfies* **(AC)** *and* $m \in S^1_{CK_\infty}(\mathbf{X}^{(1)})$. *Then the embedding* $\mathcal{F} \to L^2(E; m)$ *is compact.*

**Theorem 3.2** ([2, Theorem 1.6])**.** *Suppose that* $\mathbf{X}$ *satisfies* **(AC)** *and* $m \in S^1_{CK_\infty}(\mathbf{X}^{(1)})$. *Then the* $L^2$*-semigroup* $P_t$ *is a compact operator on* $L^2(E; m)$ *and its every eigenfunction has a finely continuous Borel measurable bounded* $m$*-version. Moreover, if* $\mathbf{X}$ *satisfies* **(RSF)***, then every eigenfunction has a bounded continuous* $m$*-version.*

Let $(\mathcal{F}_e, \mathcal{E})$ be the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$. For $\nu \in S^1_{CK_\infty}(\mathbf{X})$, the Stollmann-Voigt inequality tells us

$$\int_E f(x)^2 \nu(dx) \leq \|R_0\nu\|_\infty \mathcal{E}(f, f), \quad f \in \mathcal{F}_e$$

if $\mathbf{X}$ is transient. This means that $(\mathcal{F}_e, \mathcal{E})$ is continuously embedded in $L^2(E; \nu)$.

**Corollary 3.3** ([2, Corollary 1.7])**.** *Suppose that* $\mathbf{X}$ *is transient and it satisfies* **(AC)**. *Let* $\nu \in S^1_{CK_\infty}(\mathbf{X})$. *Then* $(\mathcal{F}_e, \mathcal{E})$ *is compactly embedded in* $L^2(E; \nu)$.

Let $\lambda_2$ be the bottom of the spectrum:

$$\lambda_2 := \inf\left\{\mathcal{E}(f, f) : f \in \mathcal{F}, \int_E f^2 dm = 1\right\}.$$

A function $\phi_0$ on $E$ is called a *ground state* of the $L^2$-generator for $\mathcal{E}$ if $\phi_0 \in \mathcal{F}$, $\|\phi_0\|_2 = 1$ and $\mathcal{E}(\phi_0, \phi_0) = \lambda_2$.

**Theorem 3.4** ([2, Theorem 1.8])**.** *Suppose that* $\mathbf{X}$ *satisfies* **(AC)**, **(I)** *and* $m \in S^1_{CK_\infty}(\mathbf{X}^{(1)})$. *Then there exists a bounded ground state* $\phi_0$ *uniquely up to sign. Moreover,* $\phi_0$ *can be taken to be strictly positive on* $E$.

Finally, we state the following general theorem to construct examples which do not possess **(RSF)**, but satisfy **(AC)**.

**Theorem 3.5** ([2, Theorem 4.1, 4.2])**.** *Suppose that* $\mathbf{X}$ *is transient and possesses* **(RSF)**. *Take* $\nu \in S_1(\mathbf{X})$ *with* $\|R_0\nu\|_\infty < \infty$ *and assume* $\nu \notin S^1_{LK}(\mathbf{X})$. *If* $\nu$ *has the full quasi-support, then the time-changed process* $(\check{\mathbf{X}}, \nu)$ *does not possess* **(RSF)***, but satisfies* **(AC)**. *Under same assumptions on* $\nu$*, there exists an* $\alpha > 0$ *such that the killed process* $\mathbf{X}^{-\alpha\nu}$ *does not possess* **(RSF)***, but satisfies* **(AC)**.

We will give concrete examples during the talk.

# References


[1] Z.-Q. Chen: Gaugeability and conditional gaugeability, *Trans. Amer. Math. Soc.* **354** (2002), 4639–4679.

[2] K. Kuwae and K. Tsuchida: Green-tight measures of Kato class and compact embedding theorem for symmetric Markov processes, preprint, 2019.

[3] M. Takeda: Compactness of symmetric Markov semi-groups and boundedness of eigenfunctions, *Trans. Amer. Math. Soc.* **372** (2019), 3905–3920.