Implicit Euler–Maruyama scheme for radial Dunkl processes

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Goal

Numerical scheme for SDE

$$\mathrm{d}X(t) = \mathrm{d}B(t) + \sum_{\alpha \in V} k \frac{\alpha}{\langle \alpha, X(t) \rangle} \mathrm{d}t, \ \ t \in [0, T],$$

$$X(0) = x \in \mathbb{W} := \{x \in \mathbb{R}^d \ ; \ \langle \alpha, x \rangle > 0, \ \forall \alpha \in V\}.$$

$k \geq 1/2$, $V = R_+ \subset \mathbb{R}^d$: positive root system.
$\mathbb{W}$: Wely chamber
Examples. Bessel proc., Dyson BM, ...

Outline

Root systems

Root systems

- A *root system* R in $\mathbb{R}^d$ is a finite set of nonzero vectors in $\mathbb{R}^d$ s.t.
  (R1) $R \cap \{c\alpha \ ; \ c \in \mathbb{R}\} = \{\alpha, -\alpha\}$, for any $\alpha \in R$
  (R2) $\sigma_\alpha(R) = R$ for any $\alpha \in R$,

  where orthogonal reflection w.r.t. α

$$\sigma_\alpha(x) = x - \frac{2\langle \alpha, x\rangle}{|\alpha|^2}\alpha, \ x \in \mathbb{R}^d.$$

- A root system R is said to be *crystallographic* if it holds that
  (R3) $c_{\alpha\beta} := 2\langle \alpha, \beta\rangle/|\alpha|^2 \in \mathbb{Z}$, for any $\alpha, \beta \in R$.
- The concept is fundamental in the theory of Lie algebras and the classification of semi-simple Lie algebras.
- $W = W(R)$, sub-group of $O(d)$: Weyl group generated by R : $W := \langle \sigma_\alpha \mid \alpha \in R\rangle$.
- The const. k can be generalized to multiplicity func.: $\exists w \in W$ s.t. $w\alpha = \beta \Rightarrow k(\alpha) = k(\beta)$.
- Let $>$ be a total ordering of $\mathbb{R}^d$. $R_+ := \{\alpha \in R; \alpha > 0\} = \{\alpha \in R; \langle \alpha, u_0\rangle > 0\}$, $\exists u_0 \in \mathbb{R}^d$.

Examples

Type A

Bessel processes ($A_1 = B_1 = C_1$):
$R := \{\pm 1\}$, $R_+ = \{+1\}$

$$\mathrm{d}X(t) = \mathrm{d}B(t) + k\frac{1}{X(t)}\mathrm{d}t.$$

Type A_{d-1}(Dyson's BM):
$R := \{e_i - e_j \in \mathbb{R}^d \ ; \ i \neq j\} \subset \{x \in \mathbb{R}^d; \sum_{i=1}^d x_i = 0\}$ is a crystallographic root system
$R_+ = \{e_i - e_j \ ; \ i < j\}$, (Lexicographic order)
$\mathbb{W}_A = \{x \in \mathbb{R}^d \ ; \ x_1 > x_2 > \cdots > x_d\}$.

$$\mathrm{d}X_i(t) = \mathrm{d}B_i(t) + \sum_{j:j\neq i} k\frac{1}{X_i(t) - X_j(t)}\mathrm{d}t.$$

Type B_d, C_d, D_d

$R := \{e_i - e_j \;;\; i \neq j\} \cup \{\mathrm{sign}(j-i)(e_i + e_j) \;;\; i \neq j\} \cup \{\pm r e_i \;;\; i = 1, \ldots, d\} \subset \mathbb{R}^d$, $r = 1, 2, 0$ are crystallographic root systems.
$R_+ = \{e_i - e_j \;;\; i < j\} \cup \{e_i + e_j \;;\; i < j\} \cup \{r e_i \;;\; i = 1, \ldots, d\}$,
$\mathbb{W}_B = \{x \in \mathbb{R}^d \;;\; x_1 > x_2 > \cdots > x_d > 0\}$
$\mathbb{W}_C = \{x \in \mathbb{R}^d \;;\; x_1 > x_2 > \cdots > x_d > 0\}$
$\mathbb{W}_D = \{x \in \mathbb{R}^d \;;\; x_1 > x_2 > \cdots > |x_d| > 0\}$.

$$\mathrm{d}X_i(t) = \mathrm{d}B_i(t) + \frac{rk}{X_i(s)}\mathrm{d}t + \sum_{j:j\neq i} k\left\{\frac{1}{X_i(t) - X_j(t)} + \frac{1}{X_i(t) + X_j(t)}\right\}\mathrm{d}t,$$

Special case of a Wishart process:

$$\begin{aligned}\mathrm{d}Y_i(t) :=& \mathrm{d}|X_i(t)|^2 \\ =& 2\sqrt{Y_i(t)}\mathrm{d}B_i(t) + \{1 + 2k(d-1+r)\}\mathrm{d}t + 2k\sum_{j:j\neq i}\frac{Y_i(t) + Y_j(t)}{Y_i(t) - Y_j(t)}\mathrm{d}t.\end{aligned}$$

Type G_2 and the other types

Type G_2:
$R := \{e_i - e_j \in \mathbb{R}^d \ ; \ 1 \le i \ne j\} \cup \{\pm(2e_i - e_j - e_k \ ; \ \{i, j, k\} = \{1, 2, 3\}\} \subset \mathbb{R}^3$
is a crystallographic root system.
$R_+ = \{e_i - e_j \ ; \ 1 \le i < j \le 3\} \cup \{2e_i - e_j - e_k ; \ \{i, j, k\} = \{1, 2, 3\}\}$
$\mathbb{W}_{G_2} = \{x \in \mathbb{R}^3 \ ; \ x_1 > x_2 > x_3, \ 2x_1 > x_2 + x_3, \ 2x_2 > x_1 + x_3, \ 2x_3 > x_1 + x_2\}$.

$$dX_i(t) = dB_i(t) + k\left\{\frac{1}{X_i(t) - X_j(t)} + \frac{1}{X_i(t) - X_k(t)}\right\}dt$$
$$+ k\left\{\frac{2}{2X_i(t) - X_j(t) - X_k(t)} + \frac{-1}{2X_j(t) - X_i(t) - X_k(t)} + \frac{-1}{2X_k(t) - X_j(t) - X_i(t)}\right.$$

Other types:
crystallographic: F_4, $F_{6,7,8}$
non crystallographic: $H_{3,4}$, $I_2(m)$, $m = 5$, $m \ge 7$.

Analytical property

Theorem 1 (cf. Dunkl and Xu [1])

V finite set of non-zero vectors in $\mathbb{R}^d$. $\Delta \prod_{v \in V} \langle v, x \rangle = 0$ iff $\exists c_v \in \mathbb{R}$ for $v \in V$ s.t. $\{c_v v \,;\, v \in V\} = R_+$ for some (reduced) root system in $\mathbb{R}^d$ and no vector in V is a scalar multiple of another vector in V.

Lemma 1

$\delta(x) := \log \prod_{\alpha \in R_+} \langle \alpha, x \rangle$, $x \in \mathbb{W}$. Then

$$\Delta \delta(x) + \sum_{\alpha \in R_+} \frac{\langle \nabla \delta(x), \alpha \rangle}{\langle \alpha, x \rangle} = 0.$$

This lemma shows that $\delta(X(t))$ with $k = 1/2$ is local martingale!

[1] *Orthogonal polynomials of several variables.* Cambridge university press, second edition (2001).

Dunkl operators and Dunkl processes

Dunkl operators and Laplacian

- Dunkl operators T_i (introduced by Dunkl (1989)[2]): differential-difference operators given by

$$T_i f(x) := \frac{\partial f(x)}{\partial x_i} + \sum_{\alpha \in R_+} k \alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle},$$

- Dunkl Laplacian:

$$\Delta_k f(x) = \sum_{i=1}^{d} T_i^2 = \Delta f(x) + 2 \sum_{\alpha \in R_+} k \left\{ \frac{\langle \nabla f(x), \alpha \rangle}{\langle x, \alpha \rangle} + \frac{f(\sigma_\alpha x) - f(x)}{\langle \alpha, x \rangle^2} \right\}.$$

Application to Dunkl operators:

- Hamiltonian op. of some Calogero-Moser-Sutherland quantum mechanical systems.
- Rösler (1998)[3]: Dunkl heat eq.

$$\left(\partial_t - \frac{\Delta_k}{2} \right) u = 0, \; u(\cdot, 0) = f$$

and its heat kernel (generalized Bessel func.)

[2] *Differential-difference operators associated to reflection groups.* Trans. Amer. Math. Soc., **311**(1), 167–183, (1989).

[3] *Generalized Hermite polynomials and the heat equation for Dunkl operators.* Commun. Math. Phys, **192** 519–541, (1998).

Dunkl processes and radial Dunkl processes

Dunkl processes:

- Rösler and Voit (1998)[4]: A Dunkl proc. $Y^x = (Y_t^x)_{t\geq 0}$ in $\mathbb{R}^d$ stating from $x \in \mathbb{R}^d$, is càdlàg Markov proc. with infinitesimal generator $\frac{1}{2}\Delta_k$.
- $u(t,x) = \mathbb{E}[f(Y_t^x)]$ is a sol. of Dunkl heat eq.
- Its satisfies martingale and the scaling properties: for $c > 0$, $(\sqrt{c}Y_t^{x/\sqrt{c}})_{t\geq 0} = (Y_{ct}^x)_{t\geq 0}$ in law, by Gallardo and Yor (2005)[5].

Radial Dunkl process:

- A radian Dunkl proc, $X = (X_t)_{t\geq 0}$ is a continuous Markov proc. with infinitesimal generator $L_k^W/2$, which is defined by

$$\frac{L_k^W f(x)}{2} := \frac{\Delta f(x)}{2} + \sum_{\alpha\in R_+} k\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle},$$

- X : W-radial part of the Dunkl proc. Y : for the canonical projection $\pi : \mathbb{R}^d \to \mathbb{R}^d/W \sim \mathbb{W} := \{x \in \mathbb{R}^d \mid \langle \alpha, x\rangle > 0,\ \forall \alpha \in R_+\}$: Weyl chamber, $X = \pi(Y)$.

[4] Rösler, M. and Voit, M. *Markov processes related with Dunkl operators.* Adv, Appl. Math., **21**(4), 575–643, (1998).

[5] *Some new examples of Markov processes which enjoy the time-inversion property.* Probab. Theory Relat. Fields, **132**, 150–162, (2005).

SDE representation

Theorem 2 (Schapira (2007)[6], Demni (2009)[7])

X: radial Dunkl proc. $T_0 := \inf\{t > 0 \mid X(t) \in \partial\mathbb{W}\}$.
If $k \geq 1/2$, then $T_0 = \infty$ a.s. and $\exists B$: d-dim BM s.t. X is a unique strong solution of the SDE

$$\mathrm{d}X(t) = \mathrm{d}B(t) + \sum_{\alpha \in R_+} k \frac{\alpha}{\langle \alpha, X(t) \rangle} \mathrm{d}t.$$

Remark: It is easy to prove pathwise uniqueness. Indeed, the drift $f_k : \mathbb{W} \to \mathbb{R}^d$

$$f_k(x) = (f_{k,1}(x), \ldots, f_{k,d}(x))^\top := \sum_{\alpha \in R_+} k \frac{\alpha}{\langle \alpha, x \rangle}.$$

satisfies one-sided Lip. (monotone) condition:

$$\langle x - y, f_k(x) - f_k(y) \rangle \leq 0, \ \forall x, y \in \mathbb{W}.$$

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[6] *The Heckman–Opdam Markov processes.* Probab. Theory Related Fields, **138**(3–4), 495–519, (2007).

[7] *Radial Dunkl processes: existence, uniqueness and hitting time.* C. R. Math. Acad. Sci. Paris, Ser. I **347**, 1125–1128, (2009).

Numerical scheme for radial Dunkl processes

Explicit Euler–Maruyama

B : d-dim BM : $(\Omega, \mathcal{F}, \mathbb{P})$: prob. sp.
$\widetilde{X}^{(n)}(0) = X(0) = x \in \mathbb{W}$

$$\widetilde{X}^{(n)}(t_{\ell+1}) := \widetilde{X}^{(n)}(t_\ell) + B(t_{\ell+1}) - B(t_\ell) + \Delta t \cdot f_k(\widetilde{X}^{(n)}(t_\ell))$$

$t_\ell := \ell \Delta t$, $\ell = 0, \ldots, n$, $\Delta t := T/n$.
But

$$\widetilde{X}^{(n)}(t_1) = x + B(t_1) + \Delta t \cdot f_k(x) \sim N(x + f_k(x)\Delta t, \Delta t)$$

Hence $\mathbb{P}\left(\widetilde{X}^{(n)}(t_\ell) \in \mathbb{W}\right) \in (0, 1)$.

Implicit Euler–Maruyama

$X^{(n)}(0) := X(0) = x(0)$ and for each $\ell = 0, \ldots, n-1$, $X^{(n)}(t_{\ell+1})$ is the unique solution in $\mathbb{W}$ of the following equation:

$$X^{(n)}(t_{\ell+1}) = X^{(n)}(t_\ell) + B(t_{\ell+1}) - B(t_\ell) + \Delta t \cdot f_k(X^{(n)}(t_{\ell+1})).$$

Lemma 2

Let $a \in \mathbb{R}^d$. Then $\exists! x \in \mathbb{W}$ s.t.

$$x = a + \Delta t \sum_{\alpha \in R_+} \frac{k\alpha}{\langle \alpha, x \rangle} = a + \Delta t \cdot f_k(x),$$

Existence: Use homotopy argument:

(1) For $w \in \mathbb{W}$, $g(x) := a + \Delta t \cdot f_k(x) - w$, $x \in \mathbb{W}$.

(2) Prove global existence of eq:

$$\begin{cases} x(t) = g(x(t)) + w + (t-1)g(w), \\ x(0) = w \end{cases} \Leftrightarrow \begin{cases} \left(I - \dfrac{\partial g(x(t))}{\partial x} \right) \dfrac{dx(t)}{dt} = g(w), \\ x(0) = w. \end{cases}$$

(3) $x(1) = g(x(1)) + w = a + \Delta t \cdot f_k(x(1))$.

Uniqueness: Apply the fact one-sided Lipschitz condition for f_k:

$$\langle x - y, f_k(x) - f_k(y) \rangle \leq 0, \ \forall x, y \in \mathbb{W}.$$

Main statements

Theorem 3 (Ngo and Taguchi)

Let R be a (reduced) root system in $\mathbb{R}^d$ and X be a radial Dunkl process with parameter k. Then

$$\mathbb{E}\left[\sup_{\ell=1,\dots,n}\left|X(t_\ell^{(n)})-X^{(n)}(t_\ell^{(n)})\right|^p\right]^{1/p}$$
$$\leq \begin{cases} \dfrac{C_p}{\sqrt{n}} & \text{if}\,,\ k\geq 7/2,\ (\Leftrightarrow 1\leq \frac{2k-1}{6}),\ p\in[1,\frac{2k-1}{6}],\cdots(A) \\ \dfrac{C_p}{n} & \text{if}\,,\ k\geq 33/2,\ (\Leftrightarrow 4\leq \frac{2k-1}{8}),\ p\in[1,\frac{2k-1}{8}],\cdots(B) \end{cases}$$

Remark 1

L^2-convergence is useful for multi-level Monte Carlo (Giles (2008)[8]):

$$\mathbb{E}[f(X)]\approx\mathbb{E}[f(X^{(2^L)})]=\sum_{\ell=1}^{L}\mathbb{E}[f(X^{(2^\ell)})-f(X^{(2^{\ell-1})})]+\mathbb{E}[f(X^{(2^0)})].$$

to reduce the computational complexity.

[8] *Multilevel monte carlo path simulation.* Oper. Res. **56**(3), 607–617, (2008).

Known results

Bessel type SDEs:

- Dereich, Neuenkirch and Szpruch (2012)[9]: (A) with $p \in [1, \frac{2k+1}{3}]$.
- Alfonsi (2013)[10]: (A) with $p \in [1, \frac{2k+1}{3}]$ and (B) with $p \in [1, \frac{2k+1}{4}]$.
- Neuenkirch and Szpruch (2014)[11]: (B) with $p \in [1, \frac{2k+1}{4}]$.

Dyson type SDE:

- Ngo and Taguchi (2019)[12]:
  (A) with $p \in [1, \frac{k-d}{3d}]$, (B) with $p \in [1, \frac{3k-d}{4d}]$.
  Ass. of k depends on the dim. d, but we can consider SDEs

$$\mathrm{d}X_i(t) = \sum_{j=1}^{d} \sigma_{i,j}(X(t))\mathrm{d}B_j(t) + \sum_{j \neq i} \frac{k}{X_i(t) - X_j(t)}\mathrm{d}t.$$

[9] *An Euler-type method for the strong approximation for the Cox-Ingersoll-Ross process.* Proc. R. Soc. A **468**, 1105–1115 (2012).

[10] *Strong order one convergence of a drift implicit Euler scheme: Application to the CIR process.* Statist. Probab. Lett. **83**(2), 602–607 (2013).

[11] *First order strong approximations of scalar SDEs defined in a domain.* Numer. Math. **128**, 103–136 (2014).

[12] *Semi-implicit Euler–Maruyama approximation for non-colliding particle systems.* To appear in Ann. Appl. Probab.

Idea of proof

Inverse moment

We need to estimate $\mathbb{E}[\langle\alpha, X(t)\rangle^{-p}]$. The idea is change of measure.

Lemma 3 (Chybiryakov 2006, PhD thesis[13], Yor 1980[14] (for Bessel case))

Let X^ν be a radial Dunkl proc with $k = \nu + \frac{1}{2} \geq \frac{1}{2}$. Define

$$Z(t) := \exp\left(M(t) - \frac{1}{2}\langle M\rangle(t)\right), \quad M(t) := \sum_{\alpha\in R_+} \nu \int_0^t \frac{\langle\alpha, \mathrm{d}B(s)\rangle}{\langle\alpha, X^0(s)\rangle}.$$

Then Z is a martingale and satisfies for any $t \in [0, T]$

$$Z(t) = \prod_{\alpha\in R_+} \frac{\langle\alpha, X^0(t)\rangle^\nu}{\langle\alpha, x(0)\rangle^\nu} \exp\left(-\frac{1}{2}\sum_{\alpha,\,\beta\in R_+}\int_0^t \frac{\nu^2\langle\alpha,\beta\rangle}{\langle\alpha, X^0(s)\rangle\langle\beta, X^0(s)\rangle}\mathrm{d}s\right). \quad (1)$$

and for $g : C([0,T]; \mathbb{W}) \to \mathbb{R}$ s.t. $\mathbb{E}[|g(X^\nu)|]$, $\mathbb{E}[|g(X^0)|Z(T)] < \infty$,

$$\mathbb{E}[g(X^\nu)] = \mathbb{E}[g(X^0)Z(T)].$$

[13] *Processus de Dunkl et relation de Lamperti.* PhD thesis, University Paris 6, (2006).

[14] *Loi de l'indice du lacet brownien, et distribution de Hartman-Watson.* Z. Wahrsch. Verw. Gebiete, **53**(1) 71–95, (1980).

Proof of Lemma 4. By Itô's formula for $\delta(x) := \log \prod_{\alpha\in R_+}\langle\alpha, x\rangle$, $x \in \mathbb{W}$,

$$\delta(X^0(t)) = \delta(x) + \int_0^t \frac{1}{2}L^W_{1/2}\delta(X^0(s))\mathrm{d}s + \sum_{\alpha\in R_+}\int_0^t \frac{\langle\alpha, \mathrm{d}B(s)\rangle}{\langle\alpha, x\rangle} = \delta(x) + \frac{1}{\nu}M(t),$$

since

$$\frac{1}{2}L^W_{1/2}\delta(x) = \frac{\Delta}{2}\delta(x) + \frac{1}{2}\sum_{\alpha\in R_+}\frac{\langle\nabla\delta(x), \alpha\rangle}{\langle\alpha, x\rangle} = 0.$$

Hence Z satisfies (1), because by def. of δ,

$$M(t) = \log\left(\prod_{\alpha\in R_+}\frac{\langle\alpha, X^0(t)\rangle^\nu}{\langle\alpha, x(0)\rangle^\nu}\right).$$

- By def, Z is a local mart.
- FACT: Z mart iff $\{Z(\tau)\ ;\ \tau : \text{stopping time} \le T\}$ is UI, (e.g. Revuz and Yor, class DL).
  We can prove UI. Indeed $\forall p > 1$, by representation of Z in (1),

$$\sup_{\tau\le T:\text{stopping time}} \mathbb{E}[|Z(\tau)|^p] \le \mathbb{E}\left[\prod_{\alpha\in R_+}\frac{|\alpha|^{p\nu}\sup_{0\le t\le T}|X^0(t)|^{p\nu}}{|\langle\alpha, x\rangle|^{p\nu}}\right] < \infty,$$

since X^0 has any positive moment (by using Itô's formula for $|x|^p$).

Lemma 4

$k = \nu + 1/2 \geq 1/2$, $\alpha \in R_+$. $\forall p \in (0, \nu]$, there exists $C_p > 0$ such that

$$\sup_{0 \leq t \leq T} \mathbb{E}[\langle \alpha, X^\nu(t) \rangle^{-p}] \leq \frac{C_p}{\prod_{\beta \in R_+} \langle \beta, x \rangle^\nu}.$$

Proof.

Apply Girsanov transform

$$\begin{aligned}
\mathbb{E}[\langle \alpha, X^\nu(t) \rangle^{-p}] &= \mathbb{E}[\langle \alpha, X^0(t) \rangle^{-p} Z(t)] \\
&\leq \mathbb{E}\left[\langle \alpha, X^0(t) \rangle^{-p} \prod_{\beta \in R_+} \frac{\langle \beta, X^0(t) \rangle^\nu}{\langle \beta, x \rangle^\nu} \right] \\
&< \infty, \text{ if } \nu - p > 0.
\end{aligned}$$

□

Proof of Theorem 3 (A)

Define

$$\begin{aligned} e(\ell+1) &:= X(t_{\ell+1}) - X^{(n)}(t_{\ell+1}) \\ &= e(\ell) + \Delta t \cdot \left\{ f_k(X(t_{\ell+1})) - f_k(X^{(n)}(t_{\ell+1}) \right\} + r(\ell), \\ r(\ell) &:= \int_{t_\ell^{(n)}}^{t_{\ell+1}} \{ f_k(X(s)) - f_k(X(t_{\ell+1})) \} \, \mathrm{d}s. \end{aligned}$$

Then

$$\begin{aligned} |e(\ell+1)|^2 &= \langle e(\ell+1), e(\ell+1) \rangle \\ &= \langle e(\ell+1), e(\ell) \rangle + \Delta t \cdot \left\langle e(\ell+1), f_k(X(t_{\ell+1})) - f_k(X^{(n)}(t_{\ell+1})) \right\rangle \\ &\quad + \langle e(\ell+1), r(\ell) \rangle \\ &\leq \frac{1}{2}|e(\ell+1)|^2 + \frac{1}{2}|e(\ell)|^2 + |e(\ell+1)||r(\ell)|. \end{aligned}$$

Hence

$$|e(\ell+1)|^2 \leq |e(\ell)|^2 + 2|e(\ell+1)||r(\ell)| \leq 2 \sum_{j=0}^{\ell} |e(j+1)||r(j)|.$$

Proof of Theorem 3 (A)

By taking the supremum,

$$\sup_{\ell=1,\dots,n} |e(\ell)|^2 \le 2\sum_{j=0}^{n-1} |e(j+1)||r(j)| \le 2 \sup_{\ell=1,\dots,n} |e(\ell)| \sum_{j=0}^{n-1} |r(j)|$$

and thus,

$$\sup_{\ell=1,\dots,n} |e(\ell)| \le 2\sum_{j=0}^{n-1} |r(j)| \le 2\int_0^T |f_k(X(s)) - f_k(X(\kappa_n(s)))| \, \mathrm{d}s.$$

where $\kappa_n(s) := t_{\ell+1} = (\ell+1)\Delta t$, if $s \in [t_\ell, t_{\ell+1})$.
Let $p \in [1, \nu/3]$. Then

$$\mathbb{E}\left[\sup_{\ell=1,\dots,m} |e(\ell)|^p\right] \le C \sum_{\alpha\in R_+} |\alpha|^p \int_0^T \mathbb{E}\left[\frac{|X(s) - X(\kappa_n(s))|^p}{\langle\alpha, X(s)\rangle^p \langle\alpha, X(\kappa_n(s))\rangle^p}\right] \mathrm{d}s$$

$$\le C \sum_{\alpha\in R_+} |\alpha|^p \int_0^T \frac{\mathbb{E}[|X(s) - X(\kappa_n(s))|^{3p}]^{1/3}}{\mathbb{E}[\langle\alpha, X(s)\rangle^{-3p}]^{1/3} \mathbb{E}[\langle\alpha, X(\kappa_n(s))\rangle^{-3p}]^{1/3}} \mathrm{d}s \le \frac{C'}{n^{p/2}},$$

bu using Kolmogorov type condition $\mathbb{E}\left[|X(t) - X(s)|^p\right] \le C_p |t-s|^{p/2}$. □

Questions, problem, future works

- How to compute the solution x of the non-linear equation

$$x = a + \Delta t \sum_{\alpha \in R_+} \frac{k\alpha}{\langle \alpha, x \rangle} \quad ?$$

  Newton's method may not work (simulation result does not converge...)

- The other idea: For $\beta \in R$, $Y_\beta(t) := \langle \beta, X(t) \rangle$.
  Then Y_β satisfies the equation

$$\mathrm{d}Y_\beta(t) = \langle \beta, \mathrm{d}B(t) \rangle + \sum_{\alpha \in R_+} k \frac{\langle \beta, \alpha \rangle}{Y_\alpha(t)} \mathrm{d}t.$$

  Moreover, define $Z_\beta(t) := Y_\beta(t)^{-1}$, then Z_β satisfies the equation

$$\mathrm{d}Z_\beta(t) = -Z_\beta(t)^2 \langle \beta, \mathrm{d}B(t) \rangle - Z_\beta(t)^2 \sum_{\alpha \in R_+} \langle \beta, \alpha \rangle Z_\alpha(t) \mathrm{d}t + Z_\beta(t)^3 |\beta|^2 \mathrm{d}t$$

  which is SDE with “super-linear growing" coefficients.

- Can we apply “tamed" EM introduced by Hutzenthaler, Jentzen, Kloeden [15] ?

[15] *Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients.* Ann. Appl. Probab., **22**(4), 1611–1641, (2012).