Implicit Euler–Maruyama scheme for radial Dunkl processes

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Abstract

Let R be a (reduced) root system in $\mathbb{R}^d$ and W be the associated reflection group. For given a vector $\xi \in \mathbb{R}^d$, the Dunkl operator T_ξ on $\mathbb{R}^d$ associated with W are introduced by Dunkl [4] and are differential-difference operators given by

$$T_\xi f(x) := \frac{\partial f(x)}{\partial \xi} + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle},$$

where $\frac{\partial}{\partial \xi}$ is the directional derivative with respect to ξ, and σ_α is the orthogonal reflection with respect to $\alpha \in \mathbb{R}^d \setminus \{0\}$, R_+ is a positive subsystem of the root system R and $k : R \to [1/2, \infty)$ is a multiplicity function. Dunkl operators have been widely studied in both mathematics and physics, for example, there operators play a crucial role to the study special functions associated with root systems and the Hamiltonian operators of some Calogero-Moser-Sutherland quantum mechanical systems. Moreover, the Dunkl Laplacian defined by $\Delta_k f(x) := \sum_{i=1}^d T_{\xi_i}^2$, for any orthonormal basis $\{\xi_1, \ldots, \xi_d\}$ of $\mathbb{R}^d$ is an important, and it has the following explicit form

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k(\alpha) \left\{ \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} + \frac{f(\sigma_\alpha x) - f(x)}{\langle \alpha, x \rangle^2} \right\}.$$

Rösler [7] studied Dunkl heat equation $(\Delta_k - \partial_t)u,\ u(\cdot, 0) = f \in C_b(\mathbb{R}^d; \mathbb{R})$ and Rösler and Voit [8] introduced Dunkl processes Y which are càdlàg Markov processes with infinitesimal generator $\Delta_k/2$ and is martingale with the scaling property. On the other hand, a radian Dunkl process $X = (X(t))_{t \geq 0}$ is a continuous Markov process with infinitesimal generator $L_k^W/2$ defined by

$$\frac{L_k^W f(x)}{2} := \frac{\Delta f(x)}{2} + \sum_{\alpha \in R_+} k(\alpha) \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle},$$

and is W-radial part of the Dunkl process Y, that is, for the canonical projection $\pi : \mathbb{R}^d \to \mathbb{R}^d/W$, $X = \pi(Y)$, as identifying the space $\mathbb{R}^d/W$ to (fundamental) Weyl chamber $\mathbb{W} := \{x \in \mathbb{R}^d\ ;\ \langle \alpha, x \rangle > 0,\ \alpha \in R_+\}$ of the root system R. Schapira [9] and Demini [2] proved that a radial Dunkl process X satisfies the following $\mathbb{W}$-valued stochastic differential equation (SDE)

$$\mathrm{d}X(t) = \mathrm{d}B(t) + \sum_{\alpha \in R_+} k(\alpha) \frac{\alpha}{\langle \alpha, X(t) \rangle} \mathrm{d}t,\ X(0) = x(0) \in \mathbb{W}, \tag{1}$$

where $B = (B(t))_{t \geq 0}$ is a d-dimensional standard Brownian motion. For example, if $R := \{\pm 1\}$ then X is a Bessel process, and for type A_{d-1} root system, that is, $R := \{e_i - e_j \in \mathbb{R}^d \ ; \ i \neq j\} \subset \{x \in \mathbb{R}^d; \sum_{i=1}^d x_i = 0\}$, then X is a Dyson's Brownian motion.

In this talks, inspired by [1, 3, 5, 6], we study a numerical analysis for radial Dunkl processes corresponding to arbitrary (reduced) root systems in $\mathbb{R}^d$, not only Bessel processes and Dyson's Brownian motions. We introduce an implicit Euler–Maruyama scheme for radial Dunkl processes (1), which takes values in the domain Wely chamber $\mathbb{W}$, and provide its rate of convergence in L^p-sup norm and path-wise sense. The key idea of the proof is to use the change of measure based on Girsanov theorem for radial Dunkl processes, which was proved in [10] for the Bessel case.

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