The strong Feller property of reflected Brownian motions on a class of planar domains

Kouhei Matsuura

Kyoto Univ.

November 18, 2019

Introduction

- A Markov operator on a topological space E is said to satisfy the strong Feller property if it maps all bounded measurable functions on E into bounded continuous functions.
- Under the strong Feller property, measure theoretic properties (of a process) are strengthened to topological ones.
- Absorbing Brownian motions always possess the strong Feller property.
- In this talk, we are concerned with the strong Feller property of reflected Brownian motions (RBMs) on general domains and continuity of the heat kernel.

Introduction

- A Markov operator on a topological space E is said to satisfy the strong Feller property if it maps all bounded measurable functions on E into bounded continuous functions.
- Under the strong Feller property, measure theoretic properties (of a process) are strengthened to topological ones.
- Absorbing Brownian motions always possess the strong Feller property.
- In this talk, we are concerned with the strong Feller property of reflected Brownian motions (RBMs) on general domains and continuity of the heat kernel.

Introduction

Let $D \subset \mathbb{R}^d$ be a domain, and m the Leb. measure on D. We define a Dirichlet form $(\mathcal{E}, H^1(D))$ by

$$H^1(D) := \{f \in L^2(D,m) \mid |\nabla f| \in L^2(D,m)\},$$
$$\mathcal{E}(f,g) := \frac{1}{2}\int_D (\nabla f, \nabla g)\, dm, \quad f,g \in H^1(D).$$

If $(\mathcal{E}, H^1(D))$ is regular on $\overline{D}$ ($H^1(D) \cap C_c(\overline{D})$ are dense in $H^1(D)$ and $C_c(\overline{D})$), it generates an m-sym. diffusion $X = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in \overline{D}})$ on $\overline{D}$.

We call X a RBM on $\overline{D}$, which may not be a semimartingale.

Question.

Under what conditions on D (or ∂D), does X have the strong Feller property?

Introduction

Let $D \subset \mathbb{R}^d$ be a domain, and m the Leb. measure on D. We define a Dirichlet form $(\mathcal{E}, H^1(D))$ by

$$H^1(D) := \{f \in L^2(D,m) \mid |\nabla f| \in L^2(D,m)\},$$
$$\mathcal{E}(f,g) := \frac{1}{2}\int_D (\nabla f, \nabla g)\, dm, \quad f,g \in H^1(D).$$

If $(\mathcal{E}, H^1(D))$ is regular on $\overline{D}$ ($H^1(D) \cap C_c(\overline{D})$ are dense in $H^1(D)$ and $C_c(\overline{D})$), it generates an m-sym. diffusion $X = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\overline{D}})$ on $\overline{D}$.

We call X a RBM on $\overline{D}$, which may not be a semimartingale.

Question.

Under what conditions on D (or ∂D), does X have the strong Feller property?

Introduction

Let $D \subset \mathbb{R}^d$ be a domain, and m the Leb. measure on D. We define a Dirichlet form $(\mathcal{E}, H^1(D))$ by

$$H^1(D) := \{f \in L^2(D,m) \mid |\nabla f| \in L^2(D,m)\},$$
$$\mathcal{E}(f,g) := \frac{1}{2}\int_D (\nabla f, \nabla g)\, dm, \quad f,g \in H^1(D).$$

If $(\mathcal{E}, H^1(D))$ is regular on $\overline{D}$ ($H^1(D) \cap C_c(\overline{D})$ are dense in $H^1(D)$ and $C_c(\overline{D})$), it generates an m-sym. diffusion $X = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\overline{D}})$ on $\overline{D}$.

We call X a RBM on $\overline{D}$, which may not be a semimartingale.

Question.

Under what conditions on D (or ∂D), does X have the strong Feller property?

Introduction

Let $D \subset \mathbb{R}^d$ be a domain, and m the Leb. measure on D. We define a Dirichlet form $(\mathcal{E}, H^1(D))$ by

$$H^1(D) := \{f \in L^2(D,m) \mid |\nabla f| \in L^2(D,m)\},$$
$$\mathcal{E}(f,g) := \frac{1}{2}\int_D (\nabla f, \nabla g)\, dm, \quad f,g \in H^1(D).$$

If $(\mathcal{E}, H^1(D))$ is regular on $\overline{D}$ ($H^1(D) \cap C_c(\overline{D})$ are dense in $H^1(D)$ and $C_c(\overline{D})$), it generates an m-sym. diffusion $X = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\overline{D}})$ on $\overline{D}$.

We call X a RBM on $\overline{D}$, which may not be a semimartingale.

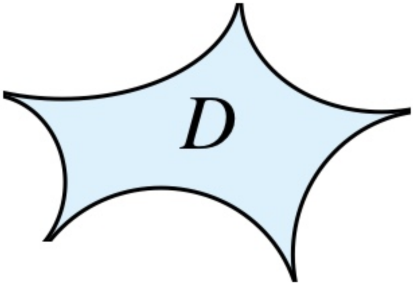
Question.

Under what conditions on D (or ∂D), does X have the strong Feller property?

Known results 1

- Bass and Hsu (1991) considered a RBM X on a bounded Lipschitz domain $D \subset \mathbb{R}^d$. The semigroup $\{P_t^X\}_{t>0}$ of X is strong Feller: $P_t^X(\mathcal{B}_b(\overline{D})) \subset C_b(\overline{D})$ for any $t > 0$.

- Fukushima and Tomisaki (1995, 1996) extended the work of Bass and Hsu. They studied a RBM on a Lipschitz domain $D \subset \mathbb{R}^d$ with cusps.



Known results 2

- **Gyrya and Saloff-Coste (2011): RBMs on uniform domains.**

  (More precisely, they considered RBMs on inner uniform domains. $(\mathcal{E}, H^1(D'))$ on an inner uniform domain D' is not necessarily regular on $\overline{D'}$.)

[Definition of uniform domains (Väisälä)]
$D \subset \mathbb{R}^d$ is unform domain if there exists $C > 0$ such that for any $x, y \in D$, there is a rectifiable curve γ in D connecting x and y with

$$\text{length}(\gamma) \leq C|x - y|,$$

and

$$\min\{|x - z|, |z - y|\} \leq C\text{dist}(z, \mathbb{R}^d \setminus D)$$

for any $z \in \gamma$.

Known results 2

- **Gyrya and Saloff-Coste (2011): RBMs on uniform domains.**

  (More precisely, they considered RBMs on inner uniform domains. $(\mathcal{E}, H^1(D'))$ on an inner uniform domain D' is not necessarily regular on $\overline{D'}$.)

[Definition of uniform domains (Väisälä)]
$D \subset \mathbb{R}^d$ is unform domain if there exists $C > 0$ such that for any $x, y \in D$, there is a rectifiable curve γ in D connecting x and y with

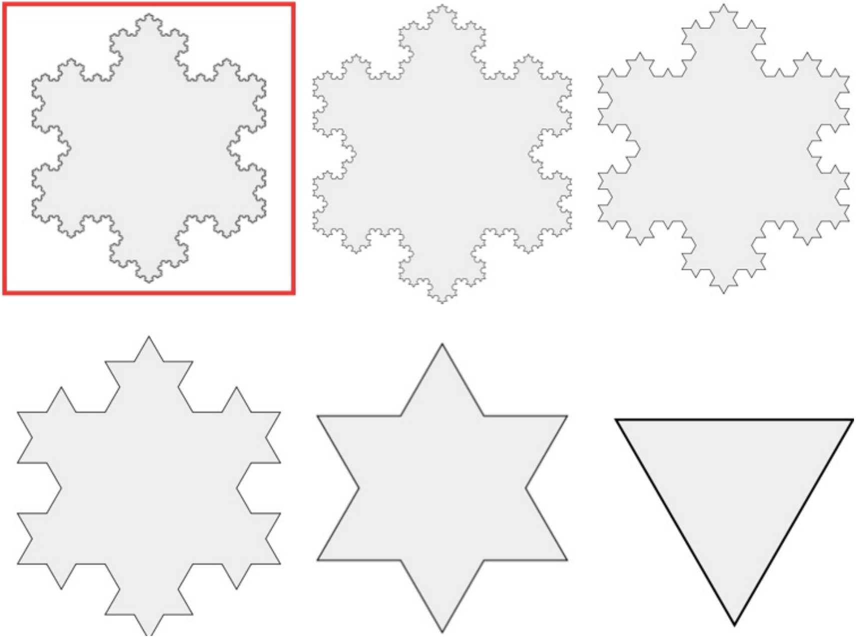
$$\text{length}(\gamma) \le C|x - y|,$$

and

$$\min\{|x - z|, |z - y|\} \le C\text{dist}(z, \mathbb{R}^d \setminus D)$$

for any $z \in \gamma$.

- **The Koch snowflake domain is a uniform domain.**



Known results 2

$\overline{D}$ is a metric space under the shortest path metric $\rho(x, y)$.

Gyrya and Saloff-Coste proved **(VD)** and **(PI)** for the Dirichlet sp. $(\overline{D}, \rho, m, \mathcal{E}, H^1(D))$.

As a result, X has a jointly conti. heat kernel $p_t^X(x, y)$.
${}^\exists c_1, c_2 \in (0, \infty)$ s.t. ${}^\forall t > 0, {}^\forall x, y \in \overline{D}$

$$p_t^X(x, y) \asymp c_1 m(B_\rho(x, \sqrt{t}))^{-1} \exp(-c_2 \rho(x, y)^2 / t).$$

${}^\exists \alpha \in (0, 1]$ s.t. the map

$$\overline{D} \ni x \longmapsto p_t^X(x, y) \in (0, \infty)$$

is α-Hölder continuous ${}^\forall y \in \overline{D}$ and ${}^\forall t > 0$.

Known results 2

$\overline{D}$ is a metric space under the shortest path metric $\rho(x, y)$.

Gyrya and Saloff-Coste proved **(VD)** and **(PI)** for the Dirichlet sp. $(\overline{D}, \rho, m, \mathcal{E}, H^1(D))$.

As a result, X has a jointly conti. heat kernel $p_t^X(x, y)$.
$^\exists c_1, c_2 \in (0, \infty)$ s.t. $^\forall t > 0, {}^\forall x, y \in \overline{D}$

$$p_t^X(x, y) \asymp c_1 m(B_\rho(x, \sqrt{t}))^{-1} \exp(-c_2 \rho(x, y)^2 / t).$$

$^\exists \alpha \in (0, 1]$ s.t. the map

$$\overline{D} \ni x \longmapsto p_t^X(x, y) \in (0, \infty)$$

is α-Hölder continuous $^\forall y \in \overline{D}$ and $^\forall t > 0$.

Known results 2

$\overline{D}$ is a metric space under the shortest path metric $\rho(x, y)$.

Gyrya and Saloff-Coste proved **(VD)** and **(PI)** for the Dirichlet sp. $(\overline{D}, \rho, m, \mathcal{E}, H^1(D))$.

As a result, X has a jointly conti. heat kernel $p_t^X(x, y)$.
$^\exists c_1, c_2 \in (0, \infty)$ s.t. $^\forall t > 0, {}^\forall x, y \in \overline{D}$

$$p_t^X(x, y) \asymp c_1 m(B_\rho(x, \sqrt{t}))^{-1} \exp(-c_2 \rho(x, y)^2 / t).$$

$^\exists \alpha \in (0, 1]$ s.t. the map

$$\overline{D} \ni x \longmapsto p_t^X(x, y) \in (0, \infty)$$

is α-Hölder continuous $^\forall y \in \overline{D}$ and $^\forall t > 0$.

Summary of main results

- **We prove the semigroup strong Feller property for RBMs on a class of planar domains.**

(1) **The class consists of Jordan domains which are images of the unit disk $\mathbb{D}$ under Hölder continuous conformal maps.**

The class is studied in the potential theory

The class $\ni$ a non-inner uniform domain

The class $\supset$ {bdd simply connected planar uniform domains}

$\ni$ the Koch snowflake domain.

(2) **On the bdd simply cnnctd planar uniform domains, the HKs of RBMs are Hölder conti.**
In this case, we give lower bounds for the Hölder exponents.

Summary of main results

- We prove the semigroup strong Feller property for RBMs on a class of planar domains.

(1) The class consists of Jordan domains which are images of the unit disk $\mathbb{D}$ under Hölder continuous conformal maps.

The class is studied in the potential theory

The class $\ni$ a non-inner uniform domain

The class $\supset$ {bdd simply connected planar uniform domains}

$\ni$ the Koch snowflake domain.

(2) On the bdd simply cnnctd planar uniform domains, the HKs of RBMs are Hölder conti.

In this case, we give lower bounds for the Hölder exponents.

Summary of main results

- We prove the semigroup strong Feller property for RBMs on a class of planar domains.

(1) The class consists of Jordan domains which are images of the unit disk $\mathbb{D}$ under Hölder continuous conformal maps.

The class is studied in the potential theory

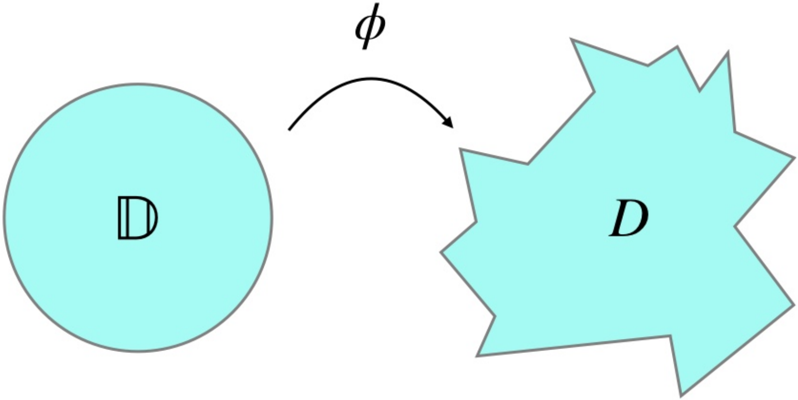
The class $\ni$ a non-inner uniform domain

The class $\supset$ {bdd simply connected planar uniform domains}

$\ni$ the Koch snowflake domain.

(2) On the bdd simply cnnctd planar uniform domains, the HKs of RBMs are Hölder conti.
In this case, we give lower bounds for the Hölder exponents.

Conformal invariance of planar RBM



Conformal invariance of planar RBM

In what follows, $D \subset \mathbb{C}$ is a Jordan domain.

Fix a conformal map $\phi : \mathbb{D} \to D$, which is extended to a homeo. $\bar{\mathbb{D}} \to \overline{D}$ ($\mathbb{D}$: the unit disk).

Let $Y = (\{Y_t\}_{t \geq 0}, \{P_y^Y\}_{y \in \bar{\mathbb{D}}})$ be the RBM on $\bar{\mathbb{D}}$.
Define $X = (\{X_t\}_{t \geq 0}, \{P_x^X\}_{x \in \overline{D}})$ as

$$P_x^X := P_{\phi^{-1}(x)}^Y, \quad x \in \overline{D},$$
$$X_t := \phi(Y_{A_t^{-1}}), \quad t \in [0.\infty),$$

where

$$A_t := \int_0^t |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds \nearrow \infty \text{ as } t \to \infty.$$

Then, the Dirichlet form of X is identified with $(\mathcal{E}, H^1(D))$ and is regular on $\overline{D}$.

Hence, X is a RBM on $\overline{D}$.

Conformal invariance of planar RBM

In what follows, $D \subset \mathbb{C}$ is a Jordan domain.

Fix a conformal map $\phi : \mathbb{D} \to D$, which is extended to a homeo. $\bar{\mathbb{D}} \to \overline{D}$ ($\mathbb{D}$: the unit disk).

Let $Y = (\{Y_t\}_{t \geq 0}, \{P_y^Y\}_{y \in \bar{\mathbb{D}}})$ be the RBM on $\bar{\mathbb{D}}$.
Define $X = (\{X_t\}_{t \geq 0}, \{P_x^X\}_{x \in \overline{D}})$ as

$$P_x^X := P_{\phi^{-1}(x)}^Y, \quad x \in \overline{D},$$
$$X_t := \phi(Y_{A_t^{-1}}), \quad t \in [0.\infty),$$

where

$$A_t := \int_0^t |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds \nearrow \infty \text{ as } t \to \infty.$$

Then, the Dirichlet form of X is identified with $(\mathcal{E}, H^1(D))$ and is regular on $\overline{D}$.

Hence, X is a RBM on $\overline{D}$.

Conformal invariance of planar RBM

In what follows, $D \subset \mathbb{C}$ is a Jordan domain.

Fix a conformal map $\phi : \mathbb{D} \to D$, which is extended to a homeo. $\bar{\mathbb{D}} \to \overline{D}$ ($\mathbb{D}$: the unit disk).

Let $Y = (\{Y_t\}_{t\geq 0}, \{P_y^Y\}_{y\in\bar{\mathbb{D}}})$ be the RBM on $\bar{\mathbb{D}}$.
Define $X = (\{X_t\}_{t\geq 0}, \{P_x^X\}_{x\in\overline{D}})$ as

$$P_x^X := P_{\phi^{-1}(x)}^Y, \quad x \in \overline{D},$$
$$X_t := \phi(Y_{A_t^{-1}}), \quad t \in [0.\infty),$$

where

$$A_t := \int_0^t |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds \nearrow \infty \text{ as } t \to \infty.$$

Then, the Dirichlet form of X is identified with $(\mathcal{E}, H^1(D))$ and is regular on $\overline{D}$.

Hence, X is a RBM on $\overline{D}$.

Conformal invariance of planar RBM

In what follows, $D \subset \mathbb{C}$ is a Jordan domain.

Fix a conformal map $\phi : \mathbb{D} \to D$, which is extended to a homeo. $\bar{\mathbb{D}} \to \overline{D}$ ($\mathbb{D}$: the unit disk).

Let $Y = (\{Y_t\}_{t\geq 0}, \{P^Y_y\}_{y\in\bar{\mathbb{D}}})$ be the RBM on $\bar{\mathbb{D}}$.
Define $X = (\{X_t\}_{t\geq 0}, \{P^X_x\}_{x\in\overline{D}})$ as

$$P^X_x := P^Y_{\phi^{-1}(x)}, \quad x \in \overline{D},$$
$$X_t := \phi(Y_{A_t^{-1}}), \quad t \in [0.\infty),$$

where

$$A_t := \int_0^t |\phi'(Y_s)|^2 \mathbf{1}_{\mathbb{D}}(Y_s)\, ds \nearrow \infty \text{ as } t \to \infty.$$

Then, the Dirichlet form of X is identified with $(\mathcal{E}, H^1(D))$ and is regular on $\overline{D}$.

Hence, X is a RBM on $\overline{D}$.

Conformal invariance of planar RBM

In what follows, $D \subset \mathbb{C}$ is a Jordan domain.

Fix a conformal map $\phi : \mathbb{D} \to D$, which is extended to a homeo. $\bar{\mathbb{D}} \to \overline{D}$ ($\mathbb{D}$: the unit disk).

Let $Y = (\{Y_t\}_{t\geq 0}, \{P_y^Y\}_{y\in\bar{\mathbb{D}}})$ be the RBM on $\bar{\mathbb{D}}$.
Define $X = (\{X_t\}_{t\geq 0}, \{P_x^X\}_{x\in\overline{D}})$ as

$$P_x^X := P_{\phi^{-1}(x)}^Y, \quad x \in \overline{D},$$
$$X_t := \phi(Y_{A_t^{-1}}), \quad t \in [0.\infty),$$

where

$$A_t := \int_0^t |\phi'(Y_s)|^2 \mathbf{1}_{\mathbb{D}}(Y_s)\, ds \nearrow \infty \text{ as } t \to \infty.$$

Then, the Dirichlet form of X is identified with $(\mathcal{E}, H^1(D))$ and is regular on $\overline{D}$.

Hence, X is a RBM on $\overline{D}$.

Main results

Denote by $\{R_\alpha^X\}_{\alpha>0}$ the resolvent of X.

Theorem 1. (M.)

Suppose that $\phi : \mathbb{D} \to D$ is κ-Hölder conti.
Then, ${}^\forall\alpha > 0$, ${}^\forall\varepsilon \in (0,\kappa)$, ${}^\exists C = C_{\alpha,\varepsilon,\kappa} > 0$ s.t.

$$\begin{aligned}&|R_\alpha^X f(x) - R_\alpha^X f(y)| \\ &\le C\|f\|_\infty |\phi^{-1}(x) - \phi^{-1}(y)|^{(\kappa-\varepsilon)}\end{aligned}$$

for ${}^\forall x, y \in \overline{D}$ and ${}^\forall f \in \mathcal{B}_b(\overline{D})$.

Denote by $\{P_t^X\}_{t>0}$ the semigroup of X.

Theorem 2. (M.)

Suppose that $\phi : \mathbb{D} \to D$ is Hölder conti. Then,
$\{P_t^X\}_{t>0}$ is strong Feller.

Main results

Denote by $\{R_\alpha^X\}_{\alpha>0}$ the resolvent of X.

Theorem 1. (M.)

Suppose that $\phi : \mathbb{D} \to D$ is κ-Hölder conti.
Then, ${}^\forall\alpha > 0$, ${}^\forall\varepsilon \in (0, \kappa)$, ${}^\exists C = C_{\alpha,\varepsilon,\kappa} > 0$ s.t.

$$\begin{aligned} &|R_\alpha^X f(x) - R_\alpha^X f(y)| \\ &\leq C\|f\|_\infty |\phi^{-1}(x) - \phi^{-1}(y)|^{(\kappa-\varepsilon)} \end{aligned}$$

for ${}^\forall x, y \in \overline{D}$ and ${}^\forall f \in \mathcal{B}_b(\overline{D})$.

Denote by $\{P_t^X\}_{t>0}$ the semigroup of X.

Theorem 2. (M.)

Suppose that $\phi : \mathbb{D} \to D$ is Hölder conti. Then, $\{P_t^X\}_{t>0}$ is strong Feller.

Main results

- If $D \subset \mathbb{R}^2$ is uniform domain, it is known that

$$\phi : \bar{\mathbb{D}} \to \overline{D} \quad \text{and} \quad \phi^{-1} : \overline{D} \to \bar{\mathbb{D}}$$

  are Hölder continuous. This means that $\{R_\alpha^X\}_{\alpha>0}$ is also Hölder continuous.

- If $D \subset \mathbb{R}^2$ is uniform domain, $\{P_t^X\}_{t>0}$ is ultracontractive.

- By a result of Bass–Kassmann–Kumagai (2010), we know

$$P_t^X f = R_\alpha^X h$$

  for some $\alpha > 0$ and $h \in L^\infty(D, m)$.

Main results

- If $D \subset \mathbb{R}^2$ is uniform domain, it is known that

$$\phi : \bar{\mathbb{D}} \to \overline{D} \quad \text{and} \quad \phi^{-1} : \overline{D} \to \bar{\mathbb{D}}$$

  are Hölder continuous. This means that $\{R_\alpha^X\}_{\alpha>0}$ is also Hölder continuous.

- If $D \subset \mathbb{R}^2$ is uniform domain, $\{P_t^X\}_{t>0}$ is ultracontractive.

- By a result of Bass–Kassmann–Kumagai (2010), we know

$$P_t^X f = R_\alpha^X h$$

  for some $\alpha > 0$ and $h \in L^\infty(D, m)$.

Main results

Corollary. (M.)

Let D be a bdd simply cnnctd planar uniform domain. Assume that

$$\phi : \overline{\mathbb{D}} \to \overline{D} \quad \text{is } \kappa\text{-Hölder continuous}$$

and

$$\phi^{-1} : \overline{D} \to \overline{\mathbb{D}} \quad \text{is } \lambda\text{-Hölder continuous.}$$

Then, ${}^{\forall}\varepsilon \in (0, \kappa), {}^{\forall}x \in \overline{D}$ and ${}^{\forall}t > 0$,

$$\overline{D} \ni y \longmapsto p_t^X(x, y)$$

is $\lambda \times (\kappa - \varepsilon)$-Hölder continuous.

Näkki–Palka (1980) showed

$$\kappa \geq \frac{2 \arcsin^2 k(\partial D)}{\pi(\pi - \arcsin k(\partial D))}, \quad \lambda \geq \frac{\pi}{2(\pi - \arcsin k(\partial D))} (> 1/2).$$

Main results

Corollary. (M.)

Let D be a bdd simply cnnctd planar uniform domain.
Assume that

$$\phi : \overline{\mathbb{D}} \to \overline{D} \quad \text{is } \kappa\text{-Hölder continuous}$$

and

$$\phi^{-1} : \overline{D} \to \overline{\mathbb{D}} \quad \text{is } \lambda\text{-Hölder continuous.}$$

Then, $^\forall \varepsilon \in (0, \kappa)$, $^\forall x \in \overline{D}$ and $^\forall t > 0$,

$$\overline{D} \ni y \longmapsto p_t^X(x, y)$$

is $\lambda \times (\kappa - \varepsilon)$-Hölder continuous.

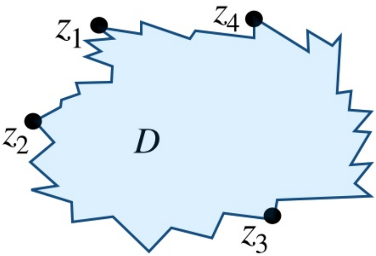
Näkki–Palka (1980) showed

$$\kappa \geq \frac{2 \arcsin^2 k(\partial D)}{\pi(\pi - \arcsin k(\partial D))}, \quad \lambda \geq \frac{\pi}{2(\pi - \arcsin k(\partial D))} (> 1/2).$$

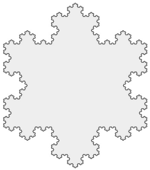
Estimates for κ and λ

$$k(\partial D) = \inf \frac{|z_1 - z_3||z_2 - z_4|}{|z_1 - z_2||z_3 - z_4| + |z_1 - z_4||z_2 - z_3|} \in (0, 1],$$

where the infimum is extended over the quadruples z_1, z_2, z_3, z_4 of finite points of Jordan arc ∂D with the property that z_1 and z_3 separate z_2 and z_4.



For the Koch snowflake,



the optimal value of $k(\partial D)$ is ...?

Outline of proof (mirror couplings of RBMs)

- **Atar and Burdzy (2004) constructed mirror couplings of RBMs on a class of Euclidean domains.**

- Let ν be the inward unit normal vector on $\partial\mathbb{D}$. The mirror coupling of RBMs (Y, Z) on $\bar{\mathbb{D}}$ is described as

$$
\begin{aligned}
Y_t &= y + B_t + \int_0^t \nu(Y_s)\, dL_s^Y, \\
Z_t &= z + W_t + \int_0^t \nu(Z_s)\, dL_s^Z, \\
W_t &= B_t - 2 \int_0^t \frac{Y_t - Z_t}{|Y_t - Z_t|^2} (Y_s - Z_s, dB_s). \\
t &< T_{\mathsf{cpl}} := \inf\{t > 0 \mid X_t = Y_t\},
\end{aligned}
$$

- W_t is the mirror image of the Brownian motion B_t w.r.t. the hyperplane M_t between Y_t and Z_t.

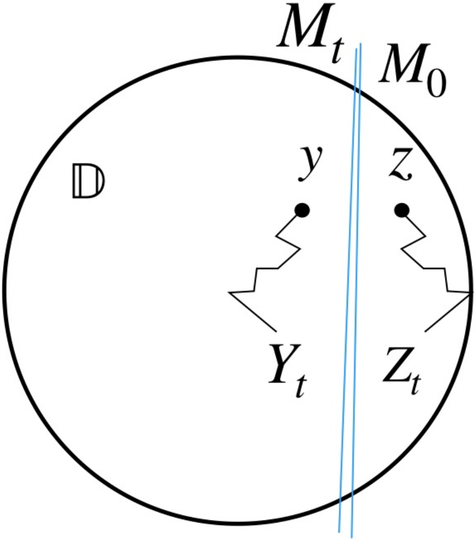
Outline of proof (mirror couplings of RBMs)

- Atar and Burdzy (2004) constructed mirror couplings of RBMs on a class of Euclidean domains.

- Let ν be the inward unit normal vector on $\partial\mathbb{D}$. The mirror coupling of RBMs (Y, Z) on $\bar{\mathbb{D}}$ is described as

$$
\begin{aligned}
Y_t &= y + B_t + \int_0^t \nu(Y_s)\, dL_s^Y, \\
Z_t &= z + W_t + \int_0^t \nu(Z_s)\, dL_s^Z, \\
W_t &= B_t - 2\int_0^t \frac{Y_t - Z_t}{|Y_t - Z_t|^2}(Y_s - Z_s, dB_s). \\
t &< T_{\mathsf{cpl}} := \inf\{t > 0 \mid X_t = Y_t\},
\end{aligned}
$$

- W_t is the mirror image of the Brownian motion B_t w.r.t. the hyperplane M_t between Y_t and Z_t.



Mirror M_t moves up to $\quad T_{\textsf{cpl}} \wedge \inf\{t > 0 \mid |X_t| = |Y_t|\}.$

- **Recall that $X = (\{X_t\}_{t\ge 0}, \{P_x^X\}_{x\in \overline{D}})$ is described as**
$$P_x^X := P^Y_{\phi^{-1}(x)}, \quad x \in \overline{D},$$
$$X_t := \phi(Y_{A_t^{-1}}), \quad t \ge 0,$$
where Y is a RBM on $\bar{\mathbb{D}}$, and $A_t = \int_0^t |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\,ds$.

- **Using the mirror coupling of RBMs (Y, Z), we have**
$$\begin{aligned}
&|R_\alpha^X f(\phi(y)) - R_\alpha^X f(\phi(z))| \\
&\le 2E_{yz}\left[\left(\int_0^{T_{\text{cpl}}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\,ds\right) \wedge \frac{1}{\alpha}\right] \\
&\quad + 2E_{yz}\left[\left(\int_0^{T_{\text{cpl}}} |\phi'(Z_s)|^2 1_{\mathbb{D}}(Z_s)\,ds\right) \wedge \frac{1}{\alpha}\right], \quad y, z \in \bar{\mathbb{D}}
\end{aligned}$$
E_{yz} is the expectation under P_{yz}.
$$\begin{aligned}
&E_{yz}\left[\left(\int_0^{T_{\text{cpl}}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\,ds\right) \wedge \frac{1}{\alpha}\right] \\
&\le E_y\left[\int_0^{\tau^Y_{B(y,r)}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\,ds\right] + \frac{1}{\alpha} \times P_{yz}(T_{\text{cpl}} > \tau^Y_{B(y,r)}).
\end{aligned}$$

- Recall that $X = (\{X_t\}_{t\geq 0}, \{P_x^X\}_{x\in\overline{D}})$ is described as
$$P_x^X := P^Y_{\phi^{-1}(x)}, \quad x \in \overline{D},$$
$$X_t := \phi(Y_{A_t^{-1}}), \quad t \geq 0,$$
where Y is a RBM on $\bar{\mathbb{D}}$, and $A_t = \int_0^t |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds$.
- Using the mirror coupling of RBMs (Y, Z), we have
$$\begin{aligned}
&|R_\alpha^X f(\phi(y)) - R_\alpha^X f(\phi(z))| \\
&\leq 2E_{yz}\left[\left(\int_0^{T_{\mathsf{cpl}}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds\right) \wedge \frac{1}{\alpha}\right] \\
&\quad + 2E_{yz}\left[\left(\int_0^{T_{\mathsf{cpl}}} |\phi'(Z_s)|^2 1_{\mathbb{D}}(Z_s)\, ds\right) \wedge \frac{1}{\alpha}\right], \quad y, z \in \bar{\mathbb{D}}
\end{aligned}$$
E_{yz} is the expectation under P_{yz}.

$$\begin{aligned}
&E_{yz}\left[\left(\int_0^{T_{\mathsf{cpl}}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds\right) \wedge \frac{1}{\alpha}\right] \\
&\leq E_y\left[\int_0^{\tau^Y_{B(y,r)}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds\right] + \frac{1}{\alpha} \times P_{yz}(T_{\mathsf{cpl}} > \tau^Y_{B(y,r)}).
\end{aligned}$$

- Recall that $X = (\{X_t\}_{t\geq 0}, \{P_x^X\}_{x\in\overline{D}})$ is described as
$$P_x^X := P_{\phi^{-1}(x)}^Y, \quad x \in \overline{D},$$
$$X_t := \phi(Y_{A_t^{-1}}), \quad t \geq 0,$$
where Y is a RBM on $\bar{\mathbb{D}}$, and $A_t = \int_0^t |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds$.
- Using the mirror coupling of RBMs (Y, Z), we have
$$\begin{aligned}
&|R_\alpha^X f(\phi(y)) - R_\alpha^X f(\phi(z))| \\
&\leq 2E_{yz}\left[\left(\int_0^{T_{\mathsf{cpl}}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds\right) \wedge \frac{1}{\alpha}\right] \\
&\quad + 2E_{yz}\left[\left(\int_0^{T_{\mathsf{cpl}}} |\phi'(Z_s)|^2 1_{\mathbb{D}}(Z_s)\, ds\right) \wedge \frac{1}{\alpha}\right], \quad y, z \in \bar{\mathbb{D}}
\end{aligned}$$
E_{yz} is the expectation under P_{yz}.
$$\begin{aligned}
&E_{yz}\left[\left(\int_0^{T_{\mathsf{cpl}}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds\right) \wedge \frac{1}{\alpha}\right] \\
&\leq E_y\left[\int_0^{\tau_{B(y,r)}^Y} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds\right] + \frac{1}{\alpha} \times P_{yz}(T_{\mathsf{cpl}} > \tau_{B(y,r)}^Y).
\end{aligned}$$

Outline of proof

$\phi : \overline{\mathbb{D}} \to \overline{D}$ is κ-Höl. continuous.

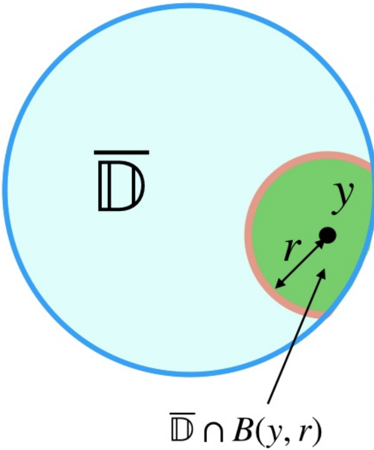
$\tau^{Y}_{B(y,r)}$
$= \inf\{t \mid Y_t \notin \overline{\mathbb{D}} \cap B(y,r)\}.$

It can be shown that

- $E^{Y}_{y}\left[\int_0^{\tau^{Y}_{B(y,r)}} |\phi'(Y_s)|^2\, ds\right]$
  $\lesssim -r^{2\kappa}\log r, \quad r \in (0, 1/32]$
  $\le (1/\varepsilon) \times r^{2\kappa-\varepsilon}, \quad \varepsilon \in (0, 2\kappa).$

- $P_{y,z}(T_{\mathsf{cpl}} > t) \lesssim |y-z|/t^{1/2},$
  $y, z \in \overline{\mathbb{D}}$

- $P^{Y}_{y}(\tau^{Y}_{B(y,r)} \le t)$
  $\lesssim \exp(-r^2/128t),\ r \ge 0,\ t > 0,$



Outline of proof

- **Setting $r = |y-z|^{1/2}$, we have**

$$I_{yz} := E_{yz}\left[\left(\int_0^{T_{\mathsf{cpl}}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\,ds\right) \wedge \frac{1}{\alpha}\right]$$

$$\leq E_y\left[\int_0^{\tau^Y_{B(y,r)}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\,ds\right] + \frac{1}{\alpha} \times P_{yz}(T_{\mathsf{cpl}} > \tau^Y_{B(y,r)})$$

$$\lesssim |y-z|^{(\kappa-\varepsilon)} + |y-z|^{1/2} \lesssim |y-z|^{(\kappa-\varepsilon)\wedge(1/2)}.$$

- By the strong Markov property, we have

$$I_{yz} = E_{yz}\left[\left(\int_0^{T_{\mathsf{cpl}}\wedge\tau^Y_{B(y,r)}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\,ds\right) \wedge \frac{1}{\alpha}\right] (\lesssim r^{2\kappa-\varepsilon})$$

$$+ E_{yz}\left[I_{Y_{\tau^Y_{B(y,r)}}, Z_{\tau^Y_{B(y,r)}}} : T_{\mathsf{cpl}} > \tau^Y_{B(y,r)}\right] \quad (\star\star\star)$$

$$\lesssim |y-z|^{(\kappa-\varepsilon)} + E_{yz}\left[|Y_{\tau^Y_{B(y,r)}} - Z_{\tau^Y_{B(y,r)}}|^{(\kappa-\varepsilon)\wedge(1/2)} : T_{\mathsf{cpl}} > \tau^Y_{B(y,r)}\right]$$

Outline of proof

- Setting $r = |y-z|^{1/2}$, we have

$$I_{yz} := E_{yz}\left[\left(\int_0^{T_{\mathsf{cpl}}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\,ds\right) \wedge \frac{1}{\alpha}\right]$$
$$\leq E_y\left[\int_0^{\tau^Y_{B(y,r)}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\,ds\right] + \frac{1}{\alpha} \times P_{yz}(T_{\mathsf{cpl}} > \tau^Y_{B(y,r)})$$
$$\lesssim |y-z|^{(\kappa-\varepsilon)} + |y-z|^{1/2} \lesssim |y-z|^{(\kappa-\varepsilon)\wedge(1/2)}.$$

- By the strong Markov property, we have

$$I_{yz} = E_{yz}\left[\left(\int_0^{T_{\mathsf{cpl}}\wedge\tau^Y_{B(y,r)}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\,ds\right) \wedge \frac{1}{\alpha}\right] (\lesssim r^{2\kappa-\varepsilon})$$
$$+ E_{yz}\left[I_{Y_{\tau^Y_{B(y,r)}}, Z_{\tau^Y_{B(y,r)}}} : T_{\mathsf{cpl}} > \tau^Y_{B(y,r)}\right] \quad (\bigstar\bigstar\bigstar)$$

$$\lesssim |y-z|^{(\kappa-\varepsilon)} + E_{yz}\left[|Y_{\tau^Y_{B(y,r)}} - Z_{\tau^Y_{B(y,r)}}|^{(\kappa-\varepsilon)\wedge(1/2)} : T_{\mathsf{cpl}} > \tau^Y_{B(y,r)}\right]$$

Outline of proof

- Setting $r = |y-z|^{1/2}$, we have

$$I_{yz} := E_{yz}\left[\left(\int_0^{T_{\mathsf{cpl}}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds\right) \wedge \frac{1}{\alpha}\right]$$
$$\leq E_y\left[\int_0^{\tau^Y_{B(y,r)}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds\right] + \frac{1}{\alpha} \times P_{yz}(T_{\mathsf{cpl}} > \tau^Y_{B(y,r)})$$
$$\lesssim |y-z|^{(\kappa-\varepsilon)} + |y-z|^{1/2} \lesssim |y-z|^{(\kappa-\varepsilon)\wedge(1/2)}.$$

- By the strong Markov property, we have

$$I_{yz} = E_{yz}\left[\left(\int_0^{T_{\mathsf{cpl}}\wedge\tau^Y_{B(y,r)}} |\phi'(Y_s)|^2 1_{\mathbb{D}}(Y_s)\, ds\right) \wedge \frac{1}{\alpha}\right] (\lesssim r^{2\kappa-\varepsilon})$$
$$+ E_{yz}\left[I_{Y_{\tau^Y_{B(y,r)}}, Z_{\tau^Y_{B(y,r)}}} : T_{\mathsf{cpl}} > \tau^Y_{B(y,r)}\right] \quad (\bigstar\bigstar\bigstar)$$

$$\lesssim |y-z|^{(\kappa-\varepsilon)} + E_{yz}\left[|Y_{\tau^Y_{B(y,r)}} - Z_{\tau^Y_{B(y,r)}}|^{(\kappa-\varepsilon)\wedge(1/2)} : T_{\mathsf{cpl}} > \tau^Y_{B(y,r)}\right]$$

- **By the Chebyshev's inequality,**

$$
\begin{aligned}
I_{yz} &\lesssim |y-z|^{(\kappa-\varepsilon)} \\
&\quad + P_{yz}\left(T_{\mathsf{cpl}} > \tau^{Y}_{B(y,r)}\right)^{1/q} \\
&\quad \times E_{yz}\left[|Y_{T_{\mathsf{cpl}}\wedge\tau^{Y}_{B(y,r)}} - Z_{T_{\mathsf{cpl}}\wedge\tau^{Y}_{B(y,r)}}|^{p\{(\kappa-\varepsilon)\wedge(1/2)\}}\right]^{1/p} \\
&\quad (\frac{1}{p}+\frac{1}{q}=1, \quad p,q\in(1,\infty)).
\end{aligned}
$$

- By Itô's formula and some geometric considerations,

$$
E_{yz}\left[\left|Y_{T_{\mathsf{cpl}}\wedge\tau^{Y}_{B(y,r)}} - Z_{T_{\mathsf{cpl}}\wedge\tau^{Y}_{B(y,r)}}\right|^{\theta}\right] \le |y-z|^{\theta}, \quad \theta\in(0,1].
$$

- Setting $p=q=2$, $\theta = p\{(\kappa-\varepsilon)\wedge(1/2)\} \le 1$, we have

$$
\begin{aligned}
I_{yz} &\lesssim |y-z|^{(\kappa-\varepsilon)} + P_{yz}\left(T_{\mathsf{cpl}} > \tau^{Y}_{B(y,r)}\right)^{1/2} \times |y-z|^{(\kappa-\varepsilon)\wedge(1/2)} \\
&\lesssim |y-z|^{(\kappa-\varepsilon)\wedge(3/4)} \quad \text{(Go back to } (\star\star\star)\text{!).}
\end{aligned}
$$

- **By the Chebyshev's inequality,**

$$\begin{aligned} I_{yz} &\lesssim |y-z|^{(\kappa-\varepsilon)} \\ &\quad + P_{yz}\left(T_{\mathsf{cpl}} > \tau^{Y}_{B(y,r)}\right)^{1/q} \\ &\quad \times E_{yz}\left[|Y_{T_{\mathsf{cpl}}\wedge\tau^{Y}_{B(y,r)}} - Z_{T_{\mathsf{cpl}}\wedge\tau^{Y}_{B(y,r)}}|^{p\{(\kappa-\varepsilon)\wedge(1/2)\}}\right]^{1/p} \\ &\quad (\frac{1}{p}+\frac{1}{q}=1, \quad p,q\in(1,\infty)). \end{aligned}$$

- **By Itô's formula and some geometric considerations,**

$$E_{yz}\left[|Y_{T_{\mathsf{cpl}}\wedge\tau^{Y}_{B(y,r)}} - Z_{T_{\mathsf{cpl}}\wedge\tau^{Y}_{B(y,r)}}|^{\theta}\right] \le |y-z|^{\theta}, \quad \theta\in(0,1].$$

- **Setting $p=q=2$, $\theta=p\{(\kappa-\varepsilon)\wedge(1/2)\}\le 1$, we have**

$$\begin{aligned} I_{yz} &\lesssim |y-z|^{(\kappa-\varepsilon)} + P_{yz}\left(T_{\mathsf{cpl}} > \tau^{Y}_{B(y,r)}\right)^{1/2} \times |y-z|^{(\kappa-\varepsilon)\wedge(1/2)} \\ &\lesssim |y-z|^{(\kappa-\varepsilon)\wedge(3/4)} \quad \text{(Go back to (★★★)!).} \end{aligned}$$

- **By the Chebyshev's inequality,**

$$I_{yz} \lesssim |y-z|^{(\kappa-\varepsilon)} + P_{yz}\left(T_{\mathsf{cpl}} > \tau^Y_{B(y,r)}\right)^{1/q} \times E_{yz}\left[|Y_{T_{\mathsf{cpl}}\wedge\tau^Y_{B(y,r)}} - Z_{T_{\mathsf{cpl}}\wedge\tau^Y_{B(y,r)}}|^{p\{(\kappa-\varepsilon)\wedge(1/2)\}}\right]^{1/p}$$

$$(\frac{1}{p}+\frac{1}{q}=1, \quad p,q\in(1,\infty)).$$

- **By Itô's formula and some geometric considerations,**

$$E_{yz}\left[|Y_{T_{\mathsf{cpl}}\wedge\tau^Y_{B(y,r)}} - Z_{T_{\mathsf{cpl}}\wedge\tau^Y_{B(y,r)}}|^{\theta}\right] \le |y-z|^{\theta}, \quad \theta\in(0,1].$$

- **Setting** $p=q=2$, $\theta = p\{(\kappa-\varepsilon)\wedge(1/2)\}\le 1$, **we have**

$$I_{yz} \lesssim |y-z|^{(\kappa-\varepsilon)} + P_{yz}\left(T_{\mathsf{cpl}} > \tau^Y_{B(y,r)}\right)^{1/2} \times |y-z|^{(\kappa-\varepsilon)\wedge(1/2)}$$

$$\lesssim |y-z|^{(\kappa-\varepsilon)\wedge(3/4)} \quad \textbf{(Go back to } (\bigstar\bigstar\bigstar)\textbf{!).}$$

Remark

Theorem 1. (M.)

Suppose that $\phi : \mathbb{D} \to D$ is κ-Hölder continuous.
Then, ${}^\forall\alpha > 0$, ${}^\forall\varepsilon \in (0,\kappa)$, ${}^\exists C = C_{\alpha,\varepsilon,\kappa} > 0$ s.t.

$$|R_\alpha^X f(x) - R_\alpha^X f(y)| \\ \leq C\|f\|_\infty |\phi^{-1}(x) - \phi^{-1}(y)|^{(\kappa-\varepsilon)}$$

for ${}^\forall x, y \in \overline{D}$ and ${}^\forall f \in \mathcal{B}_b(\overline{D})$.

- The semigroup P_t^X of X is strong Feller?
- If P_t^X is ultracontractive, there is no problem.
- There exists a non-inner uniform Jordan domain with the condition in Thm 1. It is not an extension domain.
  $E \subset \mathbb{R}^d$ is said to be an extension domain if

$$H^1(E) \subset L^p(E,m) \text{ for some } p > 2.$$

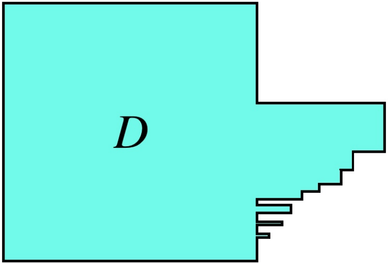
Example by Becker and Pommerenke (1982)

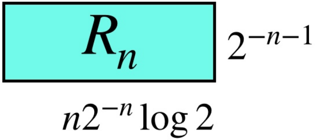
Define a Jordan domain D (which is not inner uniform) by

$$D = \{(u,v) \in \mathbb{R}^2 \mid |u| < 1,\ |v| < 1\} \cup \bigcup_{n=1}^{\infty} R_n,$$

$$R_n = \{(u,v) \in \mathbb{R}^2 \mid 0 \le u - 1 \le \frac{n \log 2}{2^n},\ |v - (1/n)| \le 2^{-n}\}.$$

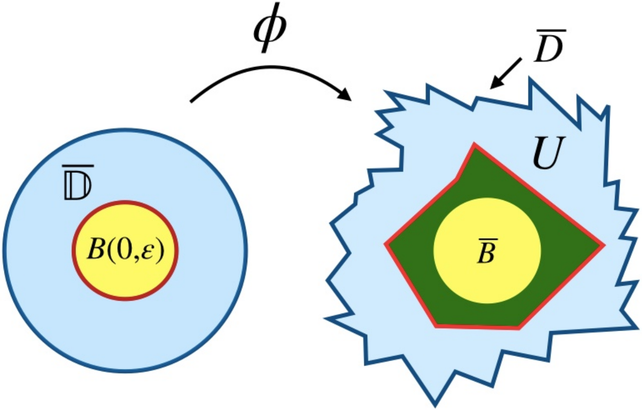
For $n \ge 5$, $R_n \cap R_{n+1} = \emptyset$.





A Refinement of Theorem 1.

- D is a domain with the condition in Theorem 1.
- Let U be an open subset of $\overline{D}$ such that $U \subset \overline{D} \setminus \overline{B}$, where $\overline{B} \subset D$ is a closed disk such that $\phi(B(0, \varepsilon)) \subset \overline{B}$
  $\mathcal{L}_U$: the Laplacian on U with the Dirichlet bdry. cond. on red line and the Neumann bdry. cond. on blue line.



A Refinement of Theorem 1.

- $G_{\overline{\mathbb{D}}\setminus\phi^{-1}(\overline{B})}(x,y) \leq 2\log(1+\varepsilon^{-1}) - 2\log|x-y|$.
- $\mathcal{L}_U$ has discrete spectrum, and $^{\exists}C_1, C_2 \in (0,\infty)$ s.t.

$$\text{(the first eigenvalue of } -\mathcal{L}_U) \geq \frac{C_1\{2\log(1+\varepsilon^{-1}) + C_2\}^{-1}}{m(U)\log\left(2 + m(U)^{-1}\right)}$$

for any $\varepsilon \in (0,1)$ and any closed disk $\overline{B} \subset D$ such that $\varphi(B(\varepsilon)) \subset B$, and any open subset U of $\overline{D}$ such that $U \subset \overline{D} \setminus B$

Lemma.

The semigroup of the part process $X^{\overline{D}\setminus\overline{B}}$ of X on $\overline{D} \setminus \overline{B}$ has a ultracontractivity.

A Refinement of Theorem 1.

- $G_{\overline{\mathbb{D}}\setminus\phi^{-1}(\overline{B})}(x,y) \leq 2\log(1+\varepsilon^{-1}) - 2\log|x-y|$.
- $\mathcal{L}_U$ has discrete spectrum, and ${}^\exists C_1, C_2 \in (0,\infty)$ s.t.

$$\text{(the first eigenvalue of } -\mathcal{L}_U) \geq \frac{C_1\{2\log(1+\varepsilon^{-1}) + C_2\}^{-1}}{m(U)\log\left(2+m(U)^{-1}\right)}$$

for any $\varepsilon \in (0,1)$ and any closed disk $\overline{B} \subset D$ such that $\varphi(B(\varepsilon)) \subset B$, and any open subset U of $\overline{D}$ such that $U \subset \overline{D} \setminus B$

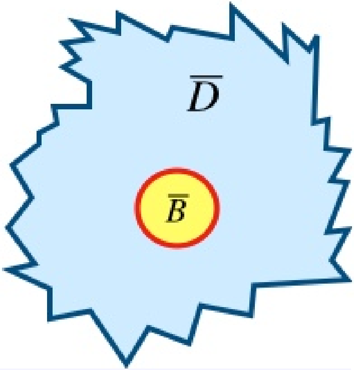
Lemma.

The semigroup of the part process $X^{\overline{D}\setminus\overline{B}}$ of X on $\overline{D} \setminus \overline{B}$ has a ultracontractivity.

A Refinement of Theorem 1.

Denote by $\{P_t^{\overline{D}\setminus\overline{B}}\}_{t>0}$ the semigroup of the part process $X^{\overline{D}\setminus\overline{B}}$ of X on $\overline{D}\setminus\overline{B}$.

- $X^{\overline{D}\setminus\overline{B}}$ is smgrp strong Feller.
- $\lim_{x\to z\in\partial B} P_t^{\overline{D}\setminus\overline{B}} f(x) = 0$ for any $t>0$ and any $f\in\mathcal{B}_b(\overline{D}\setminus\overline{B})$.



By shrinking the radius of $\overline{B}$, we have

Theorem 2. (M.)

Suppose that $\phi:\mathbb{D}\to D$ is Hölder continuous.
Then, the semigroup $\{P_t^X\}_{t>0}$ of X is strong Feller:
$P_t^X(\mathcal{B}_b(\overline{D}))\subset C_b(\overline{D})$.