The strong Feller property of reflected Brownian motions on a class of planar domains

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1 Introduction

Gyrya and Saloff-Coste [3] gave two-sided Gaussian heat kernel estimates of the Neumann heat kernels on inner uniform domains. In other words, they showed that the associated Dirichlet spaces satisfy the Poincaré inequality and the volume doubling property. As a corollary, it follows that the Neumann heat kernels are Hölder continuous. Inner uniform domains are generalized notion of uniform domains. For example, the Koch snowflake domain is a typical example of uniform domains. Thus, the boundaries of uniform domains can be fractal sets.

In this talk, we prove the semigroup strong Feller property of Neumann semigroups on a class of planar domains. Domains in the class are not necessarily inner uniform domains. Our proof is mainly based on the conformal invariance of planar reflected Brownian motions and a coupling argument. We also give quantitative lower bounds for Hölder exponents of the Neumann heat kernels on quasidisks.

2 Notation and main results

For a subset $A \subset \mathbb{C}$, we denote by $\overline{A}$ the topological closure in $\mathbb{C}$. We denote by $\mathbb{D}$ the unit disk in $\mathbb{C}$. Let $D \subset \mathbb{C}$ be a Jordan domain. Then, there exists a conformal map $\phi : \mathbb{D} \to D$, which is extended to a homeomorphism from $\overline{\mathbb{D}}$ to $\overline{D}$ by the Carathéodory's theorem. Let $X = (\{X_t\}_{t\in[0,\infty)}, \{P_x^X\}_{x\in\overline{\mathbb{D}}})$ be the reflected Brownian motion on $\overline{\mathbb{D}}$. We define a Hunt process $Y = (\{Y_t\}_{t\in[0,\infty)}, \{P_y^Y\}_{y\in\overline{D}})$ on $\overline{D}$ by

$$Y_t = \phi(X_{A_t^{-1}}),\ P_y^Y = P^X_{\phi^{-1}(y)}, \quad t \in [0,\infty),\ y \in \overline{D}.$$

Here, $\{A_t\}_{t\in[0,\infty]}$ is a positive continuous additive functional of X defined by $A_t = \int_0^t |\phi'(X_s)|^2 \mathbf{1}_D(X_s)\, ds$. Note that A_t strictly increases to ∞ as $t \to \infty$. It is easy to show that the resolvent $\{R_\alpha^Y\}_{\alpha\in(0,\infty)}$ of Y is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{C}$:

$$R_\alpha^Y f(y) = \int_{\overline{D}} r_\alpha^Y(y,z) f(z)\, dm(z), \quad y \in \overline{D},\ \alpha \in (0,\infty),\ f \in \mathcal{B}_{\mathrm{b}}(\overline{D}).$$

Here, $\mathcal{B}_{\mathrm{b}}(\overline{D})$ stands for the space of bounded measurable functions on $\overline{D}$. By [2, Examples 5.3.(2°)], the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of Y is regular on $L^2(\overline{D}, m)$, which is identified with

$$\mathcal{F} = \{f \in L^2(D,m) \mid |\nabla f| \in L^2(D,m)\}, \quad \mathcal{E}(f,g) = \frac{1}{2}\int_D \langle \nabla f, \nabla g\rangle\, dm, \quad f,g \in \mathcal{F},$$

where ∇f denotes the distributional gradient of f and $\langle \cdot,\cdot\rangle$ denotes the standard inner product on $\mathbb{C}$.

Our main theorem is as follows.

Theorem 1. Suppose that the conformal map $\phi : \mathbb{D} \to D$ is κ-Hölder continuous. Then, for any $\alpha \in (0,\infty)$ and $\varepsilon \in (0,\kappa)$, there exists a constant $C \in (0,\infty)$ such that

$$|R_\alpha^Y f(\phi(x)) - R_\alpha^Y f(\phi(y))| \le C\|f\|_{L^\infty(\overline{D},m)} |x-y|^{(\kappa-\varepsilon)\wedge(1/2)}$$

for any $x, y \in \overline{\mathbb{D}}$, and $f \in \mathcal{B}_{\mathrm{b}}(\overline{D})$. In particular, for any $\alpha \in (0,\infty)$ and $f \in \mathcal{B}_{\mathrm{b}}(\overline{D})$, $R_\alpha^Y f$ is a bounded continuous function on $\overline{D}$.

The image of the unit circle under a quasiconformal mapping on the plane is called a quasicircle. The interior of a quasicircle is called a quasidisk. It is known that quasidisks are uniform domains. Therefore, if D is a quasidisk, we can apply [3, Theorem 3.10] to show that the semigroup $\{P_t^Y\}_{t>0}$ of Y possesses a (unique) continuous kernel $p_t^Y(x,y) : (0,\infty) \times \overline{D} \times \overline{D} \to (0,\infty)$. If D is a quasidisk, it is known that the conformal map $\phi : \mathbb{D} \to D$ is bi-Hölder continuous. Then, Theorem 1 implies that the resolvent $\{R_\alpha^Y\}_{\alpha\in(0,\infty)}$ is also Hölder continuous. Combining these facts with [1, Remark 3.6], we reach the following corollary.

Corollary 1. Suppose that D is a quasidisk. Then, for each $(t,y) \in (0,\infty) \times \overline{D}$, the map $\overline{D} \ni x \mapsto p_t^Y(x,y) \in (0,\infty)$ is Hölder continuous. Furthermore, the Hölder exponent is bounded below by $\lambda\{(\kappa-\varepsilon) \wedge (1/2)\}$, where κ and λ denote the Hölder exponents of ϕ and ϕ^{-1}, respectively, and ε is an arbitrary positive number between 0 and κ.

For a Jordan curve J in $\mathbb{C}$, we define $k(J)$ as

$$k(J) = \inf \frac{|z_1 - z_3||z_2 - z_4|}{|z_1 - z_2||z_3 - z_4| + |z_1 - z_4||z_2 - z_3|} \in [0,1],$$

where the infimum is extended over the set of ordered quadruples z_1, z_2, z_3, z_4 of finite points on J with the property that z_1 and z_3 separate z_2 and z_4 on J.

A Jordan curve J in $\mathbb{C}$ is quasicircle if and only if $k(J) > 0$. In [4, Theorem 1, 2], Näkki and Palka gave estimates for Hölder exponents of conformal maps in terms of $k(J)$. Employing the result, we have

$$\kappa \geq \frac{2\arcsin^2 k(\partial D)}{\pi(\pi - \arcsin k(\partial D))}, \quad \lambda \geq \frac{\pi}{2(\pi - \arcsin k(\partial D))}.$$

in the situation of Corollary 1.

If the semigroup of Y is ultracontractive, the method of eigenfunction expansion with the resolvent strong Feller property (Theorem 1) immediately imply that the heat kernel of Y has a continuous version on $(0,\infty) \times \overline{D} \times \overline{D}$. However, there are non-inner uniform Jordan domains which satisfy the condition in Theorem 1. In this case, we do not know whether the semigroup of Y is ultracontractive. In the situation of Theorem 1, it is non-trivial even if the semigroup of Y is strong Feller. Then, we establish a Faber–Krahn type inequality for part processes of Y and obtain the following theorem.

Theorem 2. Suppose that the conformal map $\phi : \mathbb{D} \to D$ is Hölder continuous. Then, for any $t > 0$ and $f \in \mathcal{B}_{\mathrm{b}}(\overline{D})$, $P_t^Y f$ is a bounded continuous function on $\overline{D}$.

References

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