Global well-posedness of stochastic complex Ginzburg-Landau equation on the 2D torus

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1 **Introduction**

2 Ornstein-Uhlenbeck Processes

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Main theorem

In this talk, we consider the following stochastic complex Ginzburg-Landau(CGL) equation;

$$\begin{cases} \partial_t u = (i+\mu)\Delta u - \nu|u|^2 u + \lambda u + \xi \\ u(0,\cdot) = u_0. \end{cases}$$

- $u = u(t,x)$, $t \in [0,\infty)$, $x \in \mathbb{T}^2 := [-\frac{1}{2}, \frac{1}{2}]^2$.
- $i = \sqrt{-1}$, $\mu \in (0,\infty)$, $\mathrm{Re}\,\nu > 0$, $\lambda \in \mathbb{C}$.
- ξ is a complex space-time white noise;

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = 0, \ \mathbb{E}[\xi(t,x)\overline{\xi(s,y)}] = \delta(t-s)\delta(x-y).$$

Theorem

The above stochastic complex Ginzburg-Landau equation has a unique global solution for every $\mu \in (0,\infty)$.

Background: Singular SPDEs

- KPZ equation:

$$\partial_t h = \Delta h + (\partial_x h)^2 + \xi.$$

- Φ^4 equation:

$$\partial_t \Phi = \Delta \Phi - \Phi^3 + \xi.$$

Recent theories(theory of regularity structures, paracontrolled calculus and etc) provide a **local well-posedness** of many important **singular SPDEs**.
▶Local well-posedness of 3D CGL.[1]

▶ Further analysis is needed to gain more information about singular SPDEs.

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[1] M. Hoshino, Y. Inahama, and N. Naganuma. “Stochastic complex Ginzburg-Landau equation with space-time white noise”. In: *Electron. J. Probab.* 22 (2017).

- Global well-posedness:
  - Φ^4_2 equation (on the torus and in the plane).[2]
  - Φ^4_3 equation (on the torus).[3]
  - 3D CGL (on the torus) for $\mu > \frac{1}{2\sqrt{2}}$.[4]
- Markov properties:
  - Strong Feller and exponential mixing of Φ^4_2 (on the torus).[5]

►Goal: apply and develop ideas from above papers to 2D CGL.

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[2] J.-C. Mourrat and H. Weber. "Global well-posedness of the dynamic Φ^4 model in the plane". In: *Ann. Probab.* 45 (2017).

[3] J.-C. Mourrat and H. Weber. "The Dynamic Φ^4_3 Model Comes Down from Infinity". In: *Commun. Math. Phys.* 356 (2017).

[4] M. Hoshino. "Global well-posedness of complex Ginzburg–Landau equation with a space–time white noise". In: *Ann. Inst. H. Poincaré Probab. Statist.* 54 (2018).

[5] P. Tsatsoulis and H. Weber. "Spectral gap for the stochastic quantization equation on the 2-dimensional torus". In: *Ann. Inst. H. Poincaré Probab. Statist.* 54 (2018).

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Besov spaces

We fix two radial, smooth functions $\chi_{-1}, \chi : \mathbb{R}^2 \to \mathbb{R}$ such that

(i) $\mathrm{supp}(\chi_{-1}) \subset B(0, \frac{4}{3})$ and $\mathrm{supp}(\chi) \subset B(0, \frac{8}{3}) \setminus B(0, \frac{3}{4})$,

(ii) $\sum_{k=-1}^{\infty} \chi_k \equiv 1$ where $\chi_k = \chi(\cdot/2^k)$ for $k \geq 0$.

Definition

Set $\delta_k f(x) := \sum_{m \in \mathbb{Z}^2} \chi_k(m) \hat{f}(m) e^{2\pi i m \cdot x}$. For $p \in [1, \infty]$ and $\alpha \in \mathbb{R}$,

- $\|f\|_{\mathcal{B}_p^\alpha} := \sup_{k \geq -1} 2^{\alpha k} \|\delta_k f\|_{L^p}$.
- $\mathcal{B}_p^\alpha :=$ the completion of $C^\infty(\mathbb{T}^2)$ under $\|\cdot\|_{\mathcal{B}_p^\alpha}$.
- $\mathcal{C}^\alpha := \mathcal{B}_\infty^\alpha$ (generalizes the usual Hölder space).

- ***p*: integrability** ***α*: regularity**

- Schauder's estimate:
  - $\|e^{tA}f\|_{\mathcal{B}_p^\beta} \lesssim t^{-\frac{\beta-\alpha}{2}} \|f\|_{\mathcal{B}_p^\alpha}$, $A := (i+\mu)\Delta - 1$.
  - In particular, if $f \in \mathcal{C}^\alpha$ and $g(t) = \int_{-\infty}^t e^{(t-s)A} f ds$ (i.e., g is a solution of $\partial_t g = Ag + f$), we have $g(t) \in \mathcal{C}^{\alpha+\delta}$ ($\delta \in (0,2)$).
- Products:
  - $f, g \in \mathcal{C}^\alpha$, $\alpha > 0 \implies fg$ is well-defined.
  - $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$, $\alpha < 0 < \beta$, $\boldsymbol{\alpha + \beta > 0} \implies fg$ is well-defined.

$$\partial_t u = (i+\mu)\Delta u - \nu|u|^2 u + \lambda u + \xi$$

- The regularity of ξ is $-2-\varepsilon$.
- The expected regularity of u is $0-\varepsilon$.
  - $|u|^2 u$ is ill-defined.

Ornstein-Uhlenbeck processes

$A := (i+\mu)\Delta - 1$. We first consider a linear SPDE

$$\partial_t Z = AZ + \xi.$$

Heuristically, Duhamel's principle implies

$$\begin{aligned} Z(t,x) &= \int_{-\infty}^{t} e^{(t-s)A}\xi(s,x)ds \\ &= \int_{-\infty}^{t}\int_{\mathbb{T}^2} K(t-s,x-y)\xi(s,y)dyds, \end{aligned}$$

where $K(t,x) := \sum_{y\in\mathbb{Z}^2} \frac{e^{-t}}{4\pi(i+\mu)t} \exp\left(-\frac{|x-y|^2}{4(i+\mu)t}\right) \mathbb{1}_{\{t\geq 0\}}$.

- We interpret $\xi(s,y)dyds$ by complex Itô-Wiener integral.

However, $(s,y) \mapsto K(t-s,x-y)$ is not L^2-function.

Ornstein-Uhlenbeck processes

Definition

We define $Z(t) = \{Z(t,\varphi) \mid \varphi \in L^2(\mathbb{R} \times \mathbb{T}^2)\}$ by

$$Z(t,\varphi) := \int_{\mathbb{R}\times\mathbb{T}^2} \langle K(t-s,\cdot - y),\varphi\rangle_{\mathbb{T}^2}\xi(dsdy).$$

- The integral is in the sense of complex Itô-Wiener integral.
- $Z(t)$ has $\mathcal{C}^{-\alpha}$-valued modification($\forall\alpha > 0$) and $t \mapsto Z(t) \in \mathcal{C}^{-\alpha}$ is continuous.
- $Z(t)$ is a log-correlated field.
- The process $Z = \{Z(t)\}_{t\geq 0}$ is stationary ($A = (i+\mu)\Delta$ **−1**).

Wick renormalization

- $\xi_\delta(t,x)$ be a space-time mollification of ξ.
- $H_{k,l}(z;c) = z^k\overline{z}^l + \cdots$ is the (k,l)-th order complex Hermite polynomial ($H_{1,1}(z;c) = |z|^2 - c$, $H_{2,1}(z;c) = |z|^2 z - 2cz$ and so on).
- Z_δ is the solution of $\partial_t Z_\delta = AZ_\delta + \xi_\delta$ with $Z_\delta(-\infty) = 0$.

Proposition

There exists $\boldsymbol{Z^{:k,l:}}$ such that for all $\alpha, T, p \in (0,\infty)$

$$\lim_{\delta\to 0} \mathbb{E}[\sup_{0\le t\le T} \|H_{k,l}(Z_\delta(t); c_\delta) - Z^{:k,l:}(t)\|^p_{\mathcal{C}^{-\alpha}}] = 0,$$

where $c_\delta \sim \frac{1}{4\pi\mu}\log\delta^{-1}$.

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Da Prato-Debussche trick

$$\begin{cases} \partial_t u = (i+\mu)\Delta u - \nu |u|^2 u + \lambda u + \xi \\ u(0,\cdot) = u_0. \end{cases}$$

We decompose $u = Z + Y$, where Z is the Ornstein-Uhlenbeck process constructed before. Y then formally solves

$$\begin{aligned} \partial_t Y =& (i+\mu)\Delta Y - Y + (1+\lambda)(Z+Y) \\ & - \nu(|Y|^2 Y + 2Z|Y|^2 + \overline{Z} Y^2 + 2|Z|^2 Y + Z^2 \overline{Y} + |Z|^2 Z). \end{aligned}$$

- We interpret powers of Z as the corresponding Wick powers of Z.
- The expected regularity of Y is 2^- and hence the PDE for Y is well-defined.

Main theorem

We fix

$$\underline{Z} = (Z, Z^2, |Z|^2, |Z|^2 Z) \in C([0,\infty); \mathcal{C}^{-\alpha})^4.$$

and set $\Psi(Y, \underline{Z}) :=$
$(1+\lambda)(Z+Y) - \nu(|Y|^2 Y + 2Z|Y|^2 + \overline{Z} Y^2 + 2|Z|^2 Y + Z^2 \overline{Y} + |Z|^2 Z)$.
We also fix some $\alpha_1 \in (0,1)$ and $\gamma \in (0, \frac{1}{3})$.

- We say that Y is a solution of the shifted equation

$$\begin{cases} \partial_t Y = AY + \Psi(Y, \underline{Z}) & A = (i+\mu)\Delta - 1, \\ Y(0,\cdot) = Y_0 \in \mathcal{C}^{-\alpha_0} & \alpha_0 \in (0, 2/3), \end{cases}$$

if the following are satisfied;

(i) $\sup_{0<t\le T} t^{\gamma} \|Y_t\|_{\mathcal{C}^{\alpha_1}} < \infty$ for every $T \in (0,\infty)$,

(ii) $Y_t = e^{tA} Y_0 + \int_0^t e^{(t-s)A} \Psi(Y_s, \underline{Z}_s) ds$ (**mild solution**).

Main theorem

- We say that Y is a solution of

$$\begin{cases} \partial_t Y = AY + \Psi(Y, \underline{Z}) & A = (i+\mu)\Delta - 1, \\ Y(0,\cdot) = Y_0 \in \mathcal{C}^{-\alpha_0} & \alpha_0 \in (0, 2/3), \end{cases}$$

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Theorem

The shifted equation has exactly one solution over any time interval $[0, T]$.

Local well-posedness

Proposition

There exists $T^* = T^*(\|Y_0\|_{\mathcal{C}^{-\alpha_0}}, \underline{Z}) \in (0, \infty)$ *such that the shifted equation is well-posed in the interval* $[0, T^*]$.

Proof.

Set

$$B_T := \{Y : (0, T] \to \mathcal{C}^{\alpha_1} \mid \sup_{0<t\leq T} t^{\gamma}\|Y_t\|_{\mathcal{C}^{\alpha_1}} \leq 1\}$$

$$\mathcal{M}_T y(t) := e^{tA}Y_0 + \int_0^t e^{(t-s)A}\Psi(y(s), \underline{Z}_s)ds.$$

For small T^*, the map $\mathcal{M}_{T^*} : B_{T^*} \to B_{T^*}$ is a contraction. ∎

Strategy for a global solution

Proposition

There exists $T^* = T^*(\|Y_0\|_{\mathcal{C}^{-\alpha_0}}, \underline{Z}) \in (0, \infty)$ such that the shifted equation is well-posed in the interval $[0, T^*]$.

How to construct a global solution?

- We repeatedly use fixed point arguments to extend the local solution.
- Then we have to take T^* uniformly.
- **A prior bound on $\|Y_t\|_{\mathcal{C}^{-\alpha_0}}$.**
  - We need to exploit the nonlinear damping.

A priori L^p estimate

Proposition

Let $p \in [2, 2(1+\mu^2+\mu\sqrt{1+\mu^2}))$ and Y solve the shifted equation. Then we have

$$\|Y_t\|_{L^p} \lesssim_{\underline{Z}} t^{-\frac{1}{2}}$$

uniformly for the initial condition Y_0.

Proof.

The proof is similar to Mourrat and Weber(2017). We need the upper bound on p to "beat $i\Delta$ by $\mu\Delta$". ∎

- We need $p > 3$ to have the inclusion $L^p \hookrightarrow \mathcal{C}^{-\alpha_0}$. This is not the case for small μ.

Bootstrap arguments

- We obtained $\|Y_t\|_{L^p}^p \lesssim t^{-\frac{p}{2}}$.
- Actually, the previous analysis also yields

$$\|Y_t\|_{L^p}^p + \int_{t_0}^t \|Y_s\|_{L^{p+2}}^{p+2} ds + \int_{t_0}^t \|\nabla Y_s\|_{L^2}^2 ds \lesssim_{\underline{Z}} t_0^{-\frac{p}{2}}.$$

- We want to exploit the bound on the integral terms.

$$Y_t = e^{(t-t_0)A} Y_{t_0} + \int_{t_0}^t e^{(t-s)A} \Psi(Y_s, \underline{Z}_s) ds$$

► Upgrade the bounds on Y by **Shauder's estimate**.
► Key tools: **Young's convolution inequality** and **Besov embedding**.

Bootstrap arguments

We need to upgrade the L^p-bound on Y_t.

- $\int_{t_0}^{t_1} \| Y_t \|_{\mathcal{B}_{3p}^{\varepsilon}}^3 dt \lesssim t_0^{-\kappa_1}$.
- $\sup_{t_0 \le t \le t_1} \| Y_t \|_{\mathcal{B}_p^{\varepsilon}} \lesssim t_0^{-\kappa_2}$.
- $\int_{t_0}^{t_1} \| Y_t \|_{\mathcal{B}_{3p}^{2\varepsilon}}^3 dt \lesssim t_0^{-\kappa_3}$.
- $\sup_{t_0 \le t \le t_1} \| Y_t \|_{\mathcal{B}_p^{2\varepsilon}} \lesssim t_0^{-\kappa_4}$.

- We continue this process to obtain the next theorem;

Theorem

Let $\beta \in (0, 2)$. If Y solves the shifted equation, then

$$\sup_{t_0 \le t \le t_1} \| Y_t \|_{\mathcal{C}^\beta} \lesssim_{\underline{Z}, \beta} t_0^{-\kappa}$$

uniformly for the initial value Y_0.

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Markov process

The solution u of the CGL equation defines a Markov process on $\mathcal{C}^{-\alpha_0}$ (the space of the initia value).

- $P_t\Phi(x) := \mathbb{E}[\Phi(u(t))|u(0) = x]$ defines a semigroup.
- $\exists$ invariant measure by Krylov-Bogoliubov method.
- Furthermore **P_t is strong Feller** as the next theorem suggests.

Theorem

If $\|x - y\|_{\mathcal{C}^{-\alpha_0}} \leq 1$, we have

$$\|P_t^*\delta_x - P_t^*\delta_y\|_{\mathrm{TV}} \lesssim (1 + \|x\|_{\mathcal{C}^{-\alpha_0}})^\kappa \|x - y\|_{\mathcal{C}^{-\alpha_0}}^\alpha$$

for some $\kappa \in (0, \infty)$ and $\alpha \in (0, 1)$.

Key idea: **Bismut-Elworthy-Li formula**

Exponential mixing

As we add a space-time white noise, it is natural to expect exponential mixing of the CGL dynamics.

- Tsatsoulis and Weber[6] proved the exponential mixing of the Φ^4_2 equation by combining the **support theorem** and an estimate independent of the initial condition.

- Can we extend their result to 2D CGL? (supprt theorem)

[6] P. Tsatsoulis and H. Weber. "Spectral gap for the stochastic quantization equation on the 2-dimensional torus". In: *Ann. Inst. H. Poincaré Probab. Statist.* 54 (2018).

Well-posedness in the plane

- Is it possible to prove global well-posedness in the plane?
- Difficulty: We have to use **weighted Besov spaces**.
  - The work by Gubinelli and Hofmanová[7] seems useful.

[7] M. Gubinelli and M. Hofmanová. "Global Solutions to Elliptic and Parabolic Φ^4 Models in Euclidean Space". In: *Communications in Mathematical Physics* 368 (2019).

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Thank you for listening!

[7] M. Gubinelli and M. Hofmanová. "Global Solutions to Elliptic and Parabolic Φ^4 Models in Euclidean Space". In: *Communications in Mathematical Physics* 368 (2019).