Global well-posedness of stochastic complex Ginzburg-Landau equation on the 2D torus

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In this talk, we study the following stochastic complex Ginzburg-Landau equation

$$\begin{cases} \partial_t u = (i+\mu)\Delta u - \nu |u|^2 u + \lambda u + \xi, \quad t > 0,\, x \in \mathbb{T}^2, \\ u(0,\cdot) = u_0, \end{cases} \tag{1}$$

where $\mu > 0$, $\nu \in \{z \in \mathbb{C} \,|\, \Re z > 0\}$, $\lambda \in \mathbb{C}$ and $\mathbb{T}^2$ is a two dimensional torus. The random field ξ is the complex space-time white noise, i.e., the centered Gaussian random field with covariance structure

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = 0, \qquad \mathbb{E}[\xi(t,x)\overline{\xi(s,y)}] = \delta(t-s)\delta(x-y).$$

The main objective is to discuss global well-posedness of this equation and Markov properties of the solution.

The difficulty lies in the fact that a solution of the equation (1) has to be a Schwartz distribution due to the low regularity of the noise ξ, which leaves the nonlinear term $|u|^2 u$ ill-defined. To overcome this difficulty, we employ a strategy first introduced in [DD03]. Namely, we decompose a solution $u = Z + Y$, where Z is a solution of a linear SPDE

$$\partial_t Z = (i+\mu)\Delta Z - Z + \xi.$$

Although Z is still a distibution-valued random variable, we can naturally define its products via Wick renormalization.

Then Y formally solves

$$\begin{aligned}\partial_t Y &= (i+\mu)\Delta Y - Y \\ &\quad + (1+\lambda)(Z+Y) - \nu(|Y|^2 Y + 2Z|Y|^2 + \overline{Z}Y^2 + 2|Z|^2 Y + Z^2\overline{Y} + |Z|^2 Z).\end{aligned} \tag{2}$$

In the light of Shauder's estimate, Y_t has positive regularity and therefore the products appearing in the equation (2) are well-defined.

The main theorem of this talk is the following;

Theorem 1. *There exists a unique (mild) solution of* (2) *over any time interval* $[0,T]$.

For the proof of the theorem, we follow the paper [MW17], where the authors show global well-posedness of the dynamic Φ_2^4 equation. In our setting, however, the argument in [MW17] does not imply a priori L^p estimates of solutions for large p. We overcome this obstacle by bootstrap arguments, which leads to;

Theorem 2. *Let* $\beta \in (0,2)$ *and* $\mathscr{C}^\beta := \mathscr{B}^\beta_{\infty,\infty}$ *be a Besov space. Then there exists* $\kappa \in (0,\infty)$ *such that for every solution* Y *of* (2) *over* $[0,T]$,

$$\sup_{t_0 \le t \le T} \|Y_t\|_{\mathscr{C}^\beta} \lesssim_{Z,\beta,T} t_0^{-\kappa},$$

uniformly for the initial value Y_0.

Finally, I discuss my ongoing research on Markov properties of the solution as in [TW18].

References

[DD03] Giuseppe Da Prato and Arnaud Debussche. "Strong solutions to the stochastic quantization equations". In: *Ann. Probab.* 31.4 (Oct. 2003), pp. 1900–1916.

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