Stochastic quantization associated with the $\exp(\Phi)_2$-quantum field model driven by space-time white noise on the torus

(joint work with Masato Hoshino and Hiroshi Kawabi)

Seiichiro Kusuoka

Kyoto University

November 19, 2019

$\exp(\Phi)_2$-quantum field measure

Λ : the 2-dimensional torus i.e. $\Lambda = (\mathbb{R}/2\pi\mathbb{Z})^2$.

μ_0 : Nelson's free field measure on Λ with mass 1,
i.e. μ_0 is the centered Gaussian measure on $\mathcal{D}'(\Lambda)$
with the covariance $(1 - \triangle)^{-1}$.

We consider a measure:

$$\frac{1}{Z^{(\alpha)}} \exp\left(- \int_\Lambda \exp(\alpha\phi(x))dx \right) \mu_0(d\phi)$$

where $Z^{(\alpha)}$ is the normalizing constant.
Since the support of μ_0 is tempered distributions (not usual functions), $\exp(\alpha\phi)$ is not defined in usual sense.
So, we replace exp by the Wick exponential function $\exp^\diamond$.

Wick exponential function

Let $\{H_n(x); n \in \mathbb{N} \cup \{0\}\}$ be the Hermite polynomials.
For example, $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$.
For a Gaussian random variable X with mean 0, define Wick polynomials $: X^n :$ by

$$: X^n := E[X^2]^{n/2} H_n \left(E[X^2]^{-1/2} X \right).$$

Then, regarding ϕ a Gaussian random variable under μ_0, we define the Wick exponential function by

$$\exp^{\diamond}(\alpha\phi) = : \exp(\alpha\phi) : = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} : \phi^n :$$

Wick products $: \phi^n :$ and the infinite sum are well-defined for μ_0-a.e. ϕ.
We also have a formal equation

$$\exp^{\diamond}(\alpha\phi) = \exp \left(\alpha\phi - \frac{\alpha^2}{2} E^{\mu_0}[\phi(\cdot)^2] \right).$$

Note that $E^{\mu_0}[\phi(\cdot)^2] = \infty$. We discuss the rigorous definition later.

$\exp(\Phi)_2$-measure and Stochastic Quantization

Via Wick exponentials we consider the $\exp(\Phi)_2$-quantum field measure:

$$\mu_{\exp}^{(\alpha)}(d\phi) = \frac{1}{Z^{(\alpha)}} \exp\left(- \int_\Lambda \exp^\diamond(\alpha\phi)(x)dx \right) \mu_0(d\phi)$$

where $Z^{(\alpha)}$ is the normalizing constant. This model is also called the Høegh-Krohn model.

In this talk, we consider the stochastic quantization of the $\exp(\Phi)_2$-measure, which is a time-evolution having the $\exp(\Phi)_2$-measure as an invariant measure. The associated SPDE is given by

$$\text{(SQE)} \quad \partial_t \Phi_t(x) = \frac{1}{2}(\triangle - 1)\Phi_t(x) - \frac{\alpha}{2} \exp^\diamond(\alpha\Phi_t)(x) + \dot{W}_t(x), \quad x \in \Lambda$$

where $\dot{W}_t(x)$ is a white noise with parameter (t, x). This SPDE is obtained by log-derivative of $\mu_{\exp}^{(\alpha)}$.

Shifted equation

Let X be the infinite-dimensional Ornstein-Uhlenbeck process, which is the solution to

$$\partial_t X_t = \frac{1}{2}(\triangle - 1)X_t + \dot{W}_t.$$

If Φ is a solution to (SQE) in some sense, $Y := \Phi - X$ satisfies

$$(*) \quad \partial_t Y_t = \frac{1}{2}(\triangle - 1) Y_t - \frac{\alpha}{2} \exp^{\diamond}(\alpha X_t) \exp(\alpha Y_t).$$

Note that $\exp^{\diamond}(\alpha\phi) = \exp\left(\alpha\phi(x) - \frac{\alpha^2}{2} E^{\mu_0}[\phi^2]\right)$.
$(*)$ is called a shifted equation. Da Prato and Debussche (2003) solve the SPDE obtained by stochastic quantization of $P(\Phi)_2$-measures by using the shifted equation.

Relation to Φ^4-models

The Φ^4-stochastic quantization equation is given by

$$\partial_t \Phi_t(x) = \frac{1}{2}(\triangle - 1)\Phi_t(x) - (\Phi_t^3 - (C_1 + C_2)\cdot \Phi_t) + \dot{W}_t(x), \quad x \in \Lambda,$$

where C_1 and C_2 are renormalization constants ($C_1 = C_2 = \infty$).
In this case, the shifted equation is given by

$$\partial_t Y_t = \frac{1}{2}(\triangle - 1)Y_t - Y_t^3 - 3 :X_t^2: Y_t - 3X_t Y_t^2 - :X_t^3: - C_2 Y_t.$$

So, $:X_t^2:$ is the most singular coefficient. If Λ is d-dimensional torus, $:X_t^2:$ will be a $W^{2-d-\varepsilon,\infty}$-valued process.
On the other hand, $\exp^\diamond(\alpha X_t)$ $(\mathrm{dim}\Lambda = 2)$ will be $W^{-\alpha^2/(4\pi)-\varepsilon,2}$-valued.
In view of the singularities,

$$\alpha = 0 \text{ on exp-model} \sim \Phi^4\text{-model with } \mathrm{dim}\Lambda = 2$$
$$\alpha = \sqrt{4\pi} \text{ on exp-model} \sim \Phi^4\text{-model with } \mathrm{dim}\Lambda = 3.$$

Some known results

- Høegh-Krohn (1971):
  Introduction of $\exp(\Phi)_2$-model in Hamiltonian setting
- Albeverio and Høegh-Krohn (1974):
  Construction of $\exp(\Phi)_2$-measure by Euclidean quantum field theory
- Albeverio and Röckner (1991):
  Construction of the Markov process associated to (SQE) by Dirichlet forms for $|\alpha| < \sqrt{4\pi}$.
- Garban (to appear in JFA):
  Local (in time) well-posedness of (SQE) for $|\alpha| < (2\sqrt{2} - \sqrt{6}) \times \sqrt{2\pi}$ by singular SPDE methods. (Remark: $\triangle - 1$ is replaced by $\triangle$.)
- Albeverio, Kawabi, Mihalache, and Röckner (preprint):
  Regularized version is studied by Dirichlet forms.
- Oh, Robert and Wang (preprint after our work):
  Hyperbolic case is studied.

We show the global well-posedness of (SQE) for $|\alpha| < \sqrt{4\pi}$ by singular SPDE methods.

Difficulties to apply usual singular SPDE methods

It is difficult to apply a general theory (regularity structure or paracontrolled calculus) to exp-model as they are. Because

- When we apply a general theory, we usually assume that inputs (driving processes) are $W^{s,\infty}$-valued processes.
  Indeed, the coefficients of shifted equation of Φ^4-stochastic quantization equation $(: X^n :)$ are $W^{s,\infty}$-valued for suitable s (s can be negative!). On the other hand, $\exp^{\diamond}(\alpha X_t)$ is $W^{-\alpha^2/(4\pi)-\varepsilon,2}$-valued process, and to improve the integrability 2 we need to loose the regularity.
- Moreover, exp does not have polynomial growth, and the derivatives of exp are unbounded.

Hairer and Shen (2016), Chandra, Hairer and Shen (preprint) study the sine-Gordon model (the case that exp is replaced by sin). In view of singularities exp-model is same as sin-model, sin is bounded function with bounded derivatives.

Contents

Contents

Approximation operator

Let ψ be a Borel measurable function on $\mathbb{R}^2$ and assume

- $0 \leq \psi(x) \leq 1$ for $x \in \mathbb{R}^2$
- $\psi(x) = \psi(-x)$ for $x \in \mathbb{R}^2$
- $\sup_{x \in \mathbb{R}^2 \setminus \{0\}} |x|^{-\theta} |\psi(x) - 1| < \infty$ for some $\theta \in (0,1)$
  (Hölder continuity at 0)
- $\sup_{x \in \mathbb{R}^2} |x|^m |\psi(x)| < \infty$ for some $m \geq 4$.

By using ψ we define an approximation operator P_N on $\mathcal{D}'(\Lambda)$ by

$$P_N f(x) = \sum_{k \in \mathbb{Z}^2} \psi(2^{-N} k) \langle f, e_k \rangle e_k(x), \quad x \in \Lambda,$$

where $\{e_k\}$ is the Fourier basis.
For example, $\psi \in \mathcal{S}$ with $0 \leq \psi \leq 1$, $\psi = \mathbb{I}_{[-L,L]^2}$ $(L > 0)$,
$\psi = \mathbb{I}_{\{|x| \leq r\}}$ $(r > 0)$.

Approximation of Wick exponentials

Let $\psi_N := \psi(2^{-N}\cdot)$ and define approximation of Wick exponentials by

$$C_N := \int (P_N\phi(x))^2 \mu_0(d\phi) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}^2} \frac{\psi_N(k)^2}{1+|k|^2}$$

$$\exp_N^{\diamond}(\alpha\phi)(x) := \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} C_N^{n/2} H_n(C_N^{-1/2} P_N\phi(x))$$

$$= \exp\left(\alpha P_N\phi(x) - \frac{\alpha^2}{2} C_N\right).$$

Denote $W^{-\beta,2}$ by $H^{-\beta}$.

Existence of Wick exponentials

Theorem

For $|\alpha| < \sqrt{4\pi}$, let $\beta \in (\alpha^2/(4\pi), 1)$. Then $\{\exp_N^\diamond(\alpha\phi)\}$ converges in $H^{-\beta}$ for μ_0-a.e. ϕ, and in $L^2(\mu_0; H^{-\beta})$. Moreover, the limit $\exp^\diamond(\alpha\phi)$ is independent of ψ.

Proof.

Calculate

$$
\begin{aligned}
&\sum_{N=1}^\infty 2^{\varepsilon N} \sum_{k\in\mathbb{Z}^2} \frac{1}{(1+|k|^2)^\beta} \int |\langle \exp_{N+1}^\diamond(\alpha\phi) - \exp_N^\diamond(\alpha\phi), e_k\rangle|^2 \mu_0(d\phi) \\
&= \sum_{N=1}^\infty 2^{\varepsilon N} \sum_{n=1}^\infty \frac{|\alpha|^{2n}}{(n!)^2} \sum_{k\in\mathbb{Z}^2} \frac{1}{(1+|k|^2)^\beta} \\
&\quad \times \int \left| \langle C_{N+1}^{n/2} H_n(C_{N+1}^{-1/2} P_{N+1}\phi) - C_N^{n/2} H_n(C_N^{-1/2} P_N\phi), e_k\rangle \right|^2 \mu_0(d\phi) \\
&< \infty.
\end{aligned}
$$

$\exp(\Phi)_2$-quantum field

We define

$$\mu_N^{(\alpha)}(d\phi) := \frac{1}{Z_N^{(\alpha)}} \exp\left\{ - \int_\Lambda \exp_N^\diamond(\alpha\phi)(x)dx \right\} \mu_0(d\phi).$$

where $Z_N^{(\alpha)}$ is a normalizing constant.

Corollary

Let $|\alpha| < \sqrt{4\pi}$. The $\exp(\Phi)_2$-measure $\mu_{\exp}^{(\alpha)}$ is well-defined as the limit of $\{\mu_N^{(\alpha)}\}$ in weak topology, and is absolutely continuous with respect to μ_0. In particular, the support of $\mu_{\exp}^{(\alpha)}$ is in $H^{-\varepsilon}(\Lambda)$ for $\varepsilon > 0$. Moreover, the Radon-Nikodym derivatives $\left\{ d\mu_N^{(\alpha)} / d\mu_0 \right\}$ are uniformly bounded.

Proof.

Assertions follow from almost-sure convergence of $\{\exp_N^\diamond(\alpha\phi)\}$ and

$$\exp\left\{ - \int_\Lambda \exp_N^\diamond(\alpha\phi)(x)dx \right\} \leq 1.$$

Time evolution of Wick exponentials

Let $X = X(\phi)$ be the solution to

$$\begin{cases} \partial_t X_t & = \frac{1}{2}(\triangle - 1)X_t + \dot{W}_t \\ X_0 & = \phi \end{cases}$$

It is well-known that

- μ_0 is the invariant measure of X.
- X is a $W^{-1/2-\varepsilon,p}$-valued continuous process for $\varepsilon > 0$ and $p \in [1, \infty]$.

Define an approximation of Wick exponentials by

$$\mathcal{X}_t^{(\exp,N)}(\phi) = \exp_N^{\diamond}(\alpha X_t(\phi)).$$

Theorem

For $|\alpha| < \sqrt{4\pi}$, let $\beta \in (\alpha^2/(4\pi), 1)$. Then $\{\mathcal{X}_t^{(\exp,N)}\}$ converges in $L^2([0,T]; H^{-\beta})$ $\mathbb{P} \otimes \mu_0$-a.s., and in $L^2(\mathbb{P} \otimes \mu_0; H^{-\beta})$. Moreover, the limit $\mathcal{X}_t^{(\exp,\infty)}$ is independent of ψ.

Contents

Nonnegative distributions

If a distribution $\xi \in \mathcal{D}'(\Lambda)$ satisfies that $\xi(\varphi) \geq 0$ for $\varphi \in C^\infty(\Lambda; \mathbb{R}_+)$, then we call ξ nonnegative.

Theorem

For any nonnegative $\xi \in \mathcal{D}'(\Lambda)$, there exists a (nonnegative) Borel measure μ_ξ such that

$$\xi(\varphi) = \int_\Lambda \varphi(x)\mu_\xi(dx), \quad \varphi \in \mathcal{D}(\Lambda).$$

In particular, the domain of ξ is extended to $C(\Lambda)$.

Theorem (Garban (preprint))

For nonnegative $\xi \in B^{-s}_{p,r}$ and $f \in C(\Lambda)$,

$$\|f \cdot \xi\|_{B^{-s}_{p,r}} \leq C\|f\|_{C(\Lambda)}\|\xi\|_{B^{-s}_{p,r}}.$$

Uniqueness of the solution

Let $T > 0$ and

$$\mathscr{Y}_T := \{\mathcal{Y} \in L^2([0,T]; C(\Lambda) \cap H^1) \cap C([0,T]; L^2(\Lambda)); e^{\alpha \mathcal{Y}} \in L^\infty([0,T]; C(\Lambda))\}$$

Lemma

Let $\mathcal{X} \in L^2([0,T]; H_+^{-\beta})$ and $v \in H^{2-\beta}$. Then,

$$\begin{cases} \partial_t \mathcal{Y}_t &= \frac{1}{2}(\triangle - 1)\mathcal{Y}_t - \frac{\alpha}{2} e^{\alpha \mathcal{Y}_t} \mathcal{X}_t \\ \mathcal{Y}_0 &= v, \end{cases}$$

has at most one mild solution in $\mathscr{Y}_T$.

Proof.

The conditions $\mathcal{X} \in L^2([0,T]; H_+^{-\beta})$ and $v \in H^{2-\beta}$ are sufficiently nice and $\mathscr{Y}_T$ is suitable. Moreover, $y \mapsto -e^{\alpha y}$ is decreasing and $\mathcal{X}_t$ is nonnegative. These facts yield the uniqueness. □

Existence of the solution

Lemma

For any $\mathcal{X} \in L^2([0,T]; H_+^{-\beta})$ *and* $\upsilon \in H^{2-\beta}$*, there is at least one mild solution* $\Upsilon \in \mathscr{Y}_T$*. Moreover, for any* $\delta \in (0, 1-\beta)$*, there exists a constant* $C > 0$ *independent of* $\mathcal{X}$ *and* υ *such that one has the a priori estimate*

$$\|\Upsilon\|_{L^2([0,T];H^{1+\delta})\cap C([0,T];H^\delta)\cap C^{\delta/2}([0,T];L^2)} \leq C\left\{\|\upsilon\|_{H^{2-\beta}} + e^{|\alpha|\|\upsilon\|_{C(\Lambda)}}\|\mathcal{X}\|_{L^2([0,T];H^{-\beta})}\right\}.$$

Proof.

Make a uniform estimates of approximating sequence and apply compact embeddings. □

Well-posedness of the solution

From these lemmas we obtain the following.

Theorem

Let $\mathcal{X} \in L^2([0,T]; H_+^{-\beta})$ *and* $v \in H^{2-\beta}$*. Then,*

$$\begin{cases} \partial_t \mathcal{Y}_t & = \frac{1}{2}(\triangle - 1)\mathcal{Y}_t - \frac{\alpha}{2} e^{\alpha \mathcal{Y}_t} \mathcal{X}_t \\ \mathcal{Y}_0 & = v, \end{cases}$$

has a unique mild solution in $L^2([0,T]; H^{1+\delta}) \cap C([0,T]; H^{\delta})$ *for* $\delta \in (0, 1-\beta)$*. Moreover, the mapping*

$$\mathcal{S} : H^{2-\beta} \times L^2([0,T]; H_+^{-\beta}) \ni (v, \mathcal{X}) \mapsto \mathcal{Y} \in L^2([0,T]; H^{1+\delta}) \cap C([0,T]; H^{\delta})$$

is continuous.

We call the Φ obtained in the theorem the strong solution with the initial value ϕ.

Contents

Approximation by stationary processes 1

Let ρ be a nonnegative function on $\mathbb{R}^2$ and let

$$P_N f(x) = \int_{\mathbb{R}^2} 2^{2N} \rho(2^N(x-y))\tilde{f}(y)dy, \quad x \in \Lambda, \ f \in \mathcal{D}'(\Lambda),$$

where $\tilde{f}$ is the periodic extension of f to $\mathbb{R}^2$. Then the operator P_N is a nonnegative operator, i.e. $P_N f \geq 0$ if $f \geq 0$.
Let $\psi := \mathcal{F}\rho$ (the Fourier transform of ρ) and assume that ψ satisfies the conditions in a previous slide. We remark that we are able to choose usual mollifiers as P_N.
We consider the regularized $\exp(\Phi)_2$-measure by

$$\mu_N^{(\alpha)}(d\phi) := \frac{1}{Z_N^{(\alpha)}} \exp\left\{ - \int_\Lambda \exp_N^\diamond(\alpha\phi)(x)dx \right\} \mu_0(d\phi),$$

where $Z_N^{(\alpha)} > 0$ is the normalizing constant, and the SPDE associated with this measure.

Approximation by stationary processes 2

Theorem

Let $|\alpha| < \sqrt{4\pi}$ *and* P_N *as above. Let* $N \in \mathbb{N}$ *and consider the solution* $\mathbf{\Phi}^N = \mathbf{\Phi}^N(\phi)$ *of an SPDE*

$$\begin{cases} \partial_t \mathbf{\Phi}_t^N = \frac{1}{2}(\triangle - 1)\mathbf{\Phi}_t^N - \frac{\alpha}{2} P_N \exp\left(\alpha P_N \mathbf{\Phi}_t^N - \frac{\alpha^2}{2} C_N\right) + \dot{W}_t, \\ \mathbf{\Phi}_0^N = \phi \in \mathcal{D}'(\Lambda). \end{cases}$$

Let ξ_N *be a r. v. with the law* $\mu_N^{(\alpha)}$ *and independent of* W. *Then* $\bar{\mathbf{\Phi}}^N = \mathbf{\Phi}^N(\xi_N)$ *is a stationary process and the family* $\{\bar{\mathbf{\Phi}}^N\}_{N=1}^{\infty}$ *converges in law to the strong solution* $\bar{\mathbf{\Phi}}$ *with an initial law* $\mu^{(\alpha)}$, *in the space* $C([0,T]; H^{-\varepsilon}(\Lambda))$ *for any* $T > 0$. *Moreover,* $\bar{\mathbf{\Phi}}_t$ *is also stationary.*

Proof.

We show the tightness of the solutions to the shifted equations and apply the uniqueness of the limit in the previous theorem. □

Contents

Now we introduce a pre-Dirichlet form $(\mathcal{E}, \mathfrak{F}C_b^\infty)$. We fix $\beta \in (\frac{\alpha^2}{4\pi}, 1)$ and let $H = L^2(\Lambda; \mathbb{R})$ and $E = H^{-\beta}(\Lambda)$.

Let $\mathfrak{F}C_b^\infty$ be the space of all smooth cylinder functions on E. Note that $\mathfrak{F}C_b^\infty$ is dense in $L^p(\mu^{(\alpha)})$ for all $p \geq 1$.

For

$$F(\phi) = f(\langle \phi, l_1 \rangle, \dots, \langle \phi, l_n \rangle), \quad \phi \in E,$$

with $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n; \mathbb{R})$ and $l_1, \dots, l_n \in \mathrm{Span}\{e_k; k \in \mathbb{Z}^2\}$, we define the H-Fréchet derivative $D_H F : E \to H$ by

$$D_H F(\phi) := \sum_{j=1}^n \partial_j f\left(\langle \phi, l_1 \rangle, \dots, \langle \phi, l_n \rangle\right) l_j, \quad \phi \in E.$$

We consider a pre-Dirichlet form $(\mathcal{E}, \mathfrak{F}C_b^\infty)$ which is given by

$$\mathcal{E}(F, G) = \frac{1}{2} \int_E \left(D_H F(w), D_H G(w)\right)_H \mu^{(\alpha)}(dw), \quad F, G \in \mathfrak{F}C_b^\infty.$$

$(\mathcal{E}, \mathfrak{F}C_b^\infty)$ is closable on $L^2(\mu^{(\alpha)})$.

Identification with the one given by Dirichlet forms 2

So we can define $\mathcal{D}(\mathcal{E})$ as the completion of $\mathfrak{F}C_b^\infty$ with respect to $\mathcal{E}_1^{1/2}$-norm. By the general methods in the theory of Dirichlet forms, we have the quasi-regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and the existence of a diffusion process $\mathbb{M} = (\Theta, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, (\Psi_t)_{t\geq 0}, (\mathbb{Q}_\phi)_{\phi\in E})$ associated to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Theorem

Let $|\alpha| < \sqrt{4\pi}$. Then for $\mu^{(\alpha)}$-a.e. ϕ, Ψ coincides with the strong solution Φ driven by some $L^2(\Lambda)$-cylindrical $(\mathcal{G}_t)$-Brownian motion $\mathcal{W} = (\mathcal{W}_t)_{t\geq 0}$ with the initial value ϕ, $\mathbb{Q}_\phi$-almost surely.

Identification with the one given by Dirichlet forms 3

Proof.

It is known that Ψ satisfies

$$\langle \Psi_t, e_k \rangle = \langle \phi, e_k \rangle + \langle \mathcal{W}_t, e_k \rangle + \frac{1}{2} \int_0^t \langle \Psi_s, (\triangle - 1) e_k \rangle$$

$$- \frac{\alpha}{2} \int_0^t \langle \exp^{\diamond}(\alpha \Psi_s), e_k \rangle ds \quad \mathbb{Q}_\phi\text{-a.s.}$$

Decompose $\Psi = X(\phi) + Y$ and see

$$\mathbb{Q}_\phi \left(\exp^{\diamond}(\alpha \Psi_t) = e^{\alpha Y_t} \exp^{\diamond}(\alpha X_t) \text{ a.e. } t \right) = 1, \quad \mu^{(\alpha)}\text{-a.e. } \phi.$$

□

Contents

Summary

- General theories of singular SPDEs are not applicable (as they are) to the stochastic quantization equation associated to exp-model. We showed that the time-global solution exists uniquely.
- When $|\alpha| < \sqrt{4\pi}$, delicate problems do not occur (similarly to Φ_2^4 and different from Φ_3^4). Indeed, one subtraction $Y := \Phi - X$ is sufficient to solve the shifted equation, one renormalization constant is enough, and the exp-measure is absolutely continuous with respect to the free field measure μ_0.

  Hence, we have the uniqueness in a large class of approximations (universality).
- In proofs we use the nonnegativity of wick exponentials.

Thank you for your attention.