Stochastic quantization associated with the $\exp(\Phi)_2$-quantum field model driven by space-time white noise on the torus

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This is a joint work with Masato Hoshino and Hiroshi Kawabi. Let Λ be the 2-dimensional torus $(\mathbb{R}/2\pi\mathbb{Z})^2$ and μ_0 be Nelson's free field measure on Λ with mass 1. We consider the $\exp(\Phi)_2$-quantum field measure:

$$\mu_{\exp}^{(\alpha)}(d\phi) = \frac{1}{Z^{(\alpha)}} \exp\left(- \int_\Lambda \exp^\diamond(\alpha\phi)(x)dx \right) \mu_0(d\phi)$$

where $Z^{(\alpha)}$ is the normalizing constant and $\exp^\diamond(\alpha\phi)$ is the Wick exponential defined by

$$\exp^\diamond(\alpha\phi) = :\exp(\alpha\phi): = \sum_{n=0}^\infty \frac{\alpha^n}{n!} :\phi^n: .$$

We also have a formal equation

$$\exp^\diamond(\alpha\phi) = \exp\left(\alpha\phi - \frac{\alpha^2}{2} E^{\mu_0}[\phi(\cdot)^2] \right).$$

Here, note that $E^{\mu_0}[\phi(\cdot)^2] = \infty$. In this talk, we consider the stochastic quantization of the $\exp(\Phi)_2$-measure, which is a time-evolution having the $\exp(\Phi)_2$-measure as an invariant measure. The associated SPDE is given by

$$\partial_t \Phi_t(x) = \frac{1}{2}(\triangle - 1)\Phi_t(x) - \frac{\alpha}{2}\exp^\diamond(\alpha\Phi_t(x)) + \dot{W}_t(x), \quad x \in \Lambda \tag{1}$$

where $\dot{W}_t(x)$ is a white noise with parameter (t,x). This SPDE is obtained by log-derivative of $\mu_{\exp}^{(\alpha)}$. Let $X = X(\phi)$ be the infinite-dimensional Ornstein-Uhlenbeck process, which is the solution to

$$\begin{cases} \partial_t X_t &= \frac{1}{2}(\triangle - 1)X_t + \dot{W}_t \\ X_0 &= \phi. \end{cases}$$

If Φ is a solution to (1) in some sense, $Y := \Phi - X$ satisfies

$$\partial_t Y_t = \frac{1}{2}(\triangle - 1)Y_t - \frac{\alpha}{2}\exp^\diamond(\alpha X_t)\exp(\alpha Y_t). \tag{2}$$

We call (2) the shifted equation of (1). Via approximations for $|\alpha| < \sqrt{4\pi}$ we are able to define the process of Wick exponentials of X_t by

$$\mathcal{X}_t(\phi) = \exp^\diamond(\alpha X_t(\phi))$$

and

$$\int_{\mathcal{D}'(\Lambda)} \left(\int_0^T \|\mathcal{X}_t(\phi)\|_{H^{-\beta}(\Lambda)}^2 dt \right) \mu_0(d\phi) < \infty.$$

for $\beta \in (\alpha^2/(4\pi), 1)$. There are some difficulties to apply general theories such as the regularity structure and the paracontrolled calculus to (1). When we apply a general theory, we usually assume that inputs (driving processes) are $W^{s,\infty}$-valued processes. However, $\exp^\diamond(\alpha X_t)$ is a $W^{-\alpha^2/(4\pi)-\varepsilon,2}$-valued process, and to improve the integrability 2 we need to loose the regularity. The regularity is very serious for singular SPDEs. So, we need to solve (1) in the spaces with integrability 2 by using a model dependent argument. Another difficulty is that exponential functions do not have polynomial growth, and the derivatives are unbounded. These are the reasons that we study the $\exp(\Phi)_2$-model by using the method by singuar SPDEs. We remark that $\mathcal{X}_t(\phi)$ is a nonnegative distribution almost every ϕ, and that the nonnegativity plays an important role in the proofs of main theorems.

Theorem 1. *Assume that* $|\alpha| < \sqrt{4\pi}$ *and let* $\beta \in (\alpha^2/(4\pi), 1)$, $\mathcal{X} \in L^2([0,T]; H_+^{-\beta})$ *and* $v \in H^{2-\beta}$. *Then,*

$$\begin{cases} \partial_t \mathcal{Y}_t &= \frac{1}{2}(\triangle - 1)\mathcal{Y}_t - \frac{\alpha}{2} e^{\alpha \mathcal{Y}_t} \mathcal{X}_t \\ \mathcal{Y}_0 &= v, \end{cases}$$

has a unique mild solution in $L^2([0,T]; H^{1+\delta}) \cap C([0,T]; H^\delta)$ *for* $\delta \in (0, 1-\beta)$. *Moreover, the mapping*

$$\mathcal{S} : H^{2-\beta} \times L^2([0,T]; H_+^{-\beta}) \ni (v, \mathcal{X}) \mapsto \mathcal{Y} \in L^2([0,T]; H^{1+\delta}) \cap C([0,T]; H^\delta)$$

is continuous.

In this talk we call $\Phi := Y + X$ the strong solution to (1), where Y is the solution appeared in Theorem 1 and X is the Ornstein-Uhlenbeck process. We are able to show that the strong solution corresponds to the limit of the stationary processes associated to approximating equations to (1).

Theorem 2. *Let* $|\alpha| < \sqrt{4\pi}$, $\varepsilon > 0$, *and* P_N *as above. Let* $N \in \mathbb{N}$ *and consider the solution* $\bar{\Phi}^N$ *to*

$$\begin{cases} \partial_t \bar{\Phi}_t^N = \dfrac{1}{2}(\triangle - 1)\bar{\Phi}_t^N - \dfrac{\alpha}{2} P_N \exp\left(\alpha P_N \bar{\Phi}_t^N - \dfrac{\alpha^2}{2} C_N\right) + \dot{W}_t, \\ \bar{\Phi}_0^N = \xi_N \in \mathcal{D}'(\Lambda) \end{cases} \tag{3}$$

where ξ_N *be a random variable with the law* $\mu_N^{(\alpha)}$ *and independent of* W. *Then* $\bar{\Phi}^N$ *is a stationary process and converges in law to the strong solution with an initial law* $\mu_{\exp}^{(\alpha)}$, *in the space* $C([0,T]; H^{-\varepsilon}(\Lambda))$ *for any* $T > 0$. *Moreover, the law of* $\bar{\boldsymbol{\Phi}}_t$ *is* $\mu_{\exp}^{(\alpha)}$ *for any* $t \geq 0$.

In [1], the Markov process $\mathbb{M} = (\Theta, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, (\Psi_t)_{t\geq 0}, (Q_\phi)_{\phi \in E})$ associated to (1) by the Dirichlet form theory. As follows, we obtain the identification between the processes obtained in Theorem 1 and [1].

Theorem 3. *Let* $|\alpha| < \sqrt{4\pi}$. *Then for* $\mu_{\exp}^{(\alpha)}$-*a.e.* ϕ, *the diffusion process* Ψ *coincides with the strong solution* Φ *of (1) driven by some* $L^2(\Lambda)$-*cylindrical* $(\mathcal{G}_t)$-*Brownian motion* $\mathcal{W} = (\mathcal{W}_t)_{t\geq 0}$ *with the initial value* ϕ, Q_ϕ-*almost surely.*

References

[1] Sergio Albeverio and Michael Röckner, Stochastic differential equations in infinite dimensions: Solutions via Dirichlet forms, Probab. Theory Relat. Fields **89** (1991), 347–386.

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