Uniqueness of Dirichlet forms related to stochastic quantization of $\exp(\Phi)_2$-measures in finite volume

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This talk is based on a (still ongoing) joint work with Sergio Albeverio (Universität Bonn), Stefan Mihalache (KPMG, Frankfurt) and Michael Röckner (Universität Bielefeld). In this talk, we discuss L^p-uniqueness for the diffusion operators defined through Dirichlet forms given by space-time quantum fields with interactions of exponential type, called $\exp(\Phi)_2$-*measures* (Høegh-Krohn's model of quantum fields), in finite volume.

Let $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ be the two dimensional torus and $H^s(\mathbb{T}^2)$, $s \in \mathbb{R}$ denotes the Sobolev space of order s with periodic boundary condition. We put $H := L^2(\mathbb{T}^2)$. Let μ_0 be the mean-zero Gaussian measure on $E := H^{-\beta}(\mathbb{T}^2)$, $\beta > 0$ with covariance operator $(1-\Delta)^{-1}$. It is called the (massive) *Gaussian free field* (in finite volume). For a charge parameter $\alpha \in (-\sqrt{4\pi}, \sqrt{4\pi})$ and a Gaussian free field z, we formally introduce a random measure $\mathcal{M}_z^{(\alpha)}$ on $\mathbb{T}^2$ by

$$\mathcal{M}_z^{(\alpha)}(dx) := \exp^{\diamond}(\alpha z(x))dx = \exp\Big(\alpha z(x) - \frac{\alpha^2}{2}\mathbb{E}^{\mu_0}\big[z(x)^2\big]\Big)dx, \quad x \in \mathbb{T}^2.$$

This measure is called the the *Liouville measure* in the context of Liouville quantum gravity. We then define the $\exp(\Phi)_2$-measure $\mu = \mu_{\mathsf{exp}}^{(\alpha)}$ by

$$\mu(dz) = Z_\alpha^{-1}\exp(-\mathcal{M}_z^{(\alpha)}(\mathbb{T}^2))\mu_0(dz),$$

where $Z_\alpha > 0$ is the normalizing constant. It is a probability measure on E.

We now fix $\gamma > 0$ such that $\beta + 2\gamma > 2$, and consider a pre-Dirichlet form $(\mathcal{E}, \mathcal{FC}_b^\infty)$ which is given by

$$\mathcal{E}(F,G) = \frac{1}{2}\int_E \Big((1-\Delta)^{-\gamma}D_H F(z), D_H G(z)\Big)_H \mu(dz), \quad F, G \in \mathcal{FC}_b^\infty,$$

where $\mathcal{FC}_b^\infty$ is the set of all smooth cylindrical functions on E and D_H denotes the H-Fréchet derivative. The corresponding pre-Dirichlet operator $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$ is defined by $\mathcal{E}(F,G) = -(\mathcal{L}_0 F, G)_{L^2(\mu)}$, $F, G \in \mathcal{FC}_b^\infty$. It implies that $(\mathcal{E}, \mathcal{FC}_b^\infty)$ is closable on $L^2(\mu)$. We denote the closure of $(\mathcal{E}, \mathcal{FC}_b^\infty)$ by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Our main theorem in this talk is the following. (If time permits, I will touch a sketch of the proof based on the argument in [LR98].)

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Theorem. *Assume* $\beta, \gamma > 0$, $\beta + 2\gamma > 2$ *and* $|\alpha| < \min\{\sqrt{4\pi\gamma}, \sqrt{4\pi}\}$*. Then the pre-Dirichlet operator* $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$ *is* $L^p(\mu)$*-unique for all* $1 \le p < \frac{1}{2}\big(1 + \frac{4\pi\gamma}{\alpha^2}\big)$*. Namely, there exists exactly one* C_0*-semigroup on* $L^p(\mu)$ *such that its generator extends* $(\mathcal{L}_0, \mathcal{FC}_b^\infty)$*. In particular, we obtain the Markov uniqueness, that is, the Dirichlet form* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is the unique extension of* $(\mathcal{E}, \mathcal{FC}_b^\infty)$ *such that* $\mathcal{FC}_b^\infty$ *is contained in the domain of the associated generator.*

We remark that L^1-uniqueness for this model and L^p-uniqneness for $P(\Phi)_1$-, $\exp(\Phi)_1$-models in *infinite* volume have been obtained in [Wu00] and [AKR12], respectively.

As an application of this theorem, we have the *unique* existence of *weak* solution to the corresponding modified-stochastic quantization equation:

$$\begin{aligned}\partial_t u(t,x) = &- \frac{1}{2}(1-\Delta)^{1-\gamma}u(t,x) - \frac{\alpha}{2}(1-\Delta)^{-\gamma}\exp\Big(\alpha u(t,x) - \frac{\alpha^2}{2}\infty\Big)\\ &+ (1-\Delta)^{-\frac{\gamma}{2}}\xi(t,x), \qquad t>0,\ x\in\mathbb{T}^2,\end{aligned}$$

where $\xi = (\xi(t,x))_{t\ge 0, x\in\mathbb{T}^2}$ is a space-time white noise on $[0,\infty)\times\mathbb{T}^2$. Furthemore, by following the argument in [HKK19], we may construct a unique *strong* solution to this singular SPDE. However, it does not imply the $L^p(\mu)$-uniqueness of the Dirichlet operator. This is obvious, since a priori the latter might have extensions which generate non-Markovian semigroups which thus have no probabilistic interpretation as transition probabilities of a process. Therefore, neither of $L^p(\mu)$-uniqueness of the Dirichlet operator and strong uniqueness of the corresponding SPDE implies the other.

References

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