Paracontrolled calculus and regularity structures

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RS vs PC

Regularity structure (RS, Hairer (2014)) and paracontrolled calculus (PC, Gubinelli-Imkeller-Perkowski (2015)) both solve many singular (semilinear or quasilinear) PDEs. They are believed to be equivalent theories, but there are some gaps.

- PC is less general than RS. For example, the general KPZ equation

$$\partial_t h = \partial_x^2 h + f(h)(\partial_x h)^2 + g(h)\xi$$

  cannot be solved within PC.
- No systematic theory in PC ($\leftrightarrow$ "Black box" in RS).
- In PC, solutions are written by existing analytic tools. More informations for specific SPDEs were obtained.

$\Rightarrow$ Can we solve general SPDEs (including general KPZ) within PC?

Rough description of the main result

The first step to implant the algebraic structure of RS to PC. We obtained the equivalence between the two kinds of definitions of the solutions.

Rough path theory	RS		PC
Rough path	Model	⇔ [1]	Pararemainders
Controlled path	Modelled distribution	⇔ [2]	Paracontrolled distribution
Stochastic integral	[Bruned-Hairer -Zambotti (2019)]	Future work	No systematic theory

- [1] Bailleul-H (2018)
- [2] Bailleul-H (in preparation)

Main result

Consider a RS $\mathcal{T} = (T^+, T)$ satisfying assumptions defined later, that are satisfied in [Bruned-Hairer-Zambotti (2019)].

Theorem (Bailleul-H (2018))

The space $\mathcal{M}_{\mathrm{rap}}(\mathbb{R}^d)$ of all models rapidly decreasing at ∞ is homeomorphic to the direct product of Banach spaces;

$$\mathcal{M}_{\mathrm{rap}}(\mathbb{R}^d) \simeq \prod_{\tau \in \mathcal{B}_\circ,\, |\tau|<0} \mathcal{C}_{\mathrm{rap}}^{|\tau|}(\mathbb{R}^d) \times \prod_{\sigma \in \mathcal{P}\mathcal{B}_\circ^+} \mathcal{C}_{\mathrm{rap}}^{|\sigma|}(\mathbb{R}^d).$$

Moreover, the space $\mathcal{D}_{\mathrm{rap}}^{\gamma}(g;\mathbb{R}^d)$ of all γ-class modelled distributions rapidly decreasing at ∞ is homeomorphic to the direct product of Banach spaces;

$$\mathcal{D}_{\mathrm{rap}}^{\gamma}(g;\mathbb{R}^d) \simeq \prod_{\tau \in \mathcal{B}_\circ,\, |\tau|<\gamma} \mathcal{C}_{\mathrm{rap}}^{\gamma-|\tau|}(\mathbb{R}^d).$$

Remarks

- Original GIP's work was done in the right hand side space. Our result first show that Hairer's work and GIP's work are exactly equivalent (at least in a PDE level).
- For the space Ω^α of α-Hölder rough paths ($1/3 < \alpha \leq 1/2$), our result yields that

$$\Omega^\alpha \simeq \mathcal{C}^\alpha \times \mathcal{C}^{2\alpha}.$$

  The meaning is that, $\mathcal{C}^\alpha$ is a space where Brownian motions live, and $\mathcal{C}^{2\alpha}$ represents a difference between Itô integral and Stratonovich integral.
- For a technical reason, we use the weighted space $\mathcal{C}^\alpha_{\mathrm{rap}}$, not $\mathcal{C}^\alpha = \mathcal{B}^\alpha_{\infty,\infty}$. For any $\phi \in \mathcal{S}$ and $f \in \mathcal{C}^\alpha$,

$$\phi \cdot f \in \mathcal{C}^\alpha_{\mathrm{rap}}.$$

Related researches

- Gubinelli-Imkeller-Perkowski (2015) rewrote the reconstruction operator using harmonic analysis.
- Martin-Perkowski (2018) replaced the notion of modelled distribution by "paramodelled distribution" using GIP's operator, but the model is still needed.
- Tapia-Zambotti (2018) showed a similar result to ours for the branched rough path.

Generalization of rough path theory

(Gubinelli's) rough path theory

- $(B^i) \mapsto ((B^i), (\int B^i dB^j))$: Rough path
- $Y_t = Y_s + \sum Y_s^i (B_t^i - B_s^i) + (\text{regular enough})$: Controlled path

The space of RPs is not linear, and the space of CPs is a "fibre" on RP.

RP is interpreted as a continuous path on a Lie group, in particular, a character group on a Hopf algebra.

Regularity structure is a theory of

- Model, i.e. a continuous function taking values in a character group on a Hopf algebra.
- Modelled distribution, i.e. a "fibre" on the space of models.

The geometry is not simple.

Rough path, branched rough path, BHZ Hopf algebra

Hopf algebra H = "Jointing trees" + "Spliting a tree"
= product $(\cdot : H \otimes H \to H)$ + coproduct $(\Delta : H \to H \otimes H)$.

- Geometric RP: Words $(i_1 i_2 \dots i_n)$.

$$\begin{aligned} \text{Meaning} \quad & \textstyle\int B^i dB^j = (ij), \\ \text{Shuffle product} \quad & (ij) ⧢ (k) = (kij) + (ikj) + (ijk), \\ \text{Admissible cut} \quad & \Delta(ij) = (ij) \otimes () + (i) \otimes (j) + () \otimes (ij). \end{aligned}$$

- Branched RP: Rooted trees (Connes-Kreimer Hopf algebra (1998))

$$\int B^j B^k dB_i =$$ 

- Bruned-Hairer-Zambotti (2019): Rooted decorated trees
= Rooted tree + Taylor polynomials + Derivatives + Renormalizations.

Character group on Hopf algebra

For any Hopf algebra $(H, \cdot, \Delta)$, the set G of linear maps $g : H \to \mathbb{R}$ such that

$$g(\tau \cdot \sigma) = g(\tau)g(\sigma)$$

forms a group. The product on G:

$$(g * g')(\tau) := (g \otimes g')\Delta\tau.$$

In many settings, G is a Lie group with a canonical distance. We consider a G-valued continuous function $x \mapsto g_x$.

- α-geometric RP is a Lipschitz continuous path $t \mapsto g_t$.
- α-branched RP is a path valued in Butcher group.
- Model is Lipschitz continuous function $\mathbb{R}^d \ni x \mapsto g_x \in G$.

Concrete regularity structure

Definition

A concrete regularity structure is a pair of a graded Hopf algebra $T^+ = \bigoplus_{\alpha \in A^+} T^+_\alpha$ *with a coproduct* Δ^+ *and a graded linear space* $T = \bigoplus_{\beta \in A} T_\beta$ *with a linear map* $\Delta : T \to T \otimes T^+$ *satisfying the right comodule property on* T^+*, such that*

$$\Delta^+ \tau \in \tau \otimes \mathbf{1} + \mathbf{1} \otimes \tau + \bigoplus_{0<\beta<\alpha} T^+_\beta \otimes T^+_{\alpha-\beta},$$

$$\Delta \sigma \in \sigma \otimes \mathbf{1} + \bigoplus_{\gamma<\beta} T_\gamma \otimes T^+_{\beta-\gamma},$$

for any $\tau \in T^+_\alpha$ *and* $\sigma \in T_\beta$.

Notations

- Each $\tau \in T_\alpha^+$ (or T_α) is said to be have a homogeneity α. We write
$$|\tau| = \alpha.$$
- Fix a homogeneous basis $\mathcal{B}^{(+)}$ of $T^{(+)}$. For any $\tau, \sigma \in \mathcal{B}^{(+)}$, we define the element $\tau/\sigma \in T^+$ by
$$\Delta^{(+)}\tau = \sum_{\sigma \in \mathcal{B}^{(+)}} \sigma \otimes (\tau/\sigma).$$
- Graphical meaning is that, σ is a subtree of τ sharing a same root with τ and τ/σ is a tree $\tau/\sim_\sigma$, where $\sim_\sigma$ is an equivalence relation on the nodes of τ defined by
$$n_1 \sim_\sigma n_2 \quad \Leftrightarrow \quad n_1, n_2 \in \sigma.$$

Polynomial regularity structure

An important example is the polynomial regularity structure.

- $T^+ = T = \mathbb{R}[X_1, \dots, X_d]$.
- $X^k := \prod_{i=1}^d X_i^{k_i}$, where $k = (k_i)_{i=1}^d \in \mathbb{N}^d$.
- Product $X^k \cdot X^\ell = X^{k+\ell}$.
- Coproduct $\Delta X^k = \sum \binom{k}{\ell} X^\ell \otimes X^{k-\ell}$.

Model

Definition

The space $\mathcal{M}_{\mathrm{rap}}(\mathbb{R}^d)$ *consists of all pairs* (g, Π) *such that*

- g *is a Lipschitz continuous map from* $\mathbb{R}^d$ *to the character group* G*.*
- Π *is a continuous operator from* T *to* $\mathcal{S}'(\mathbb{R}^d)$ *such that*

  $$\Pi_x \tau := (\Pi \otimes g_x^{-1})\Delta\tau$$

  has a $\mathcal{C}^{|\tau|}$ *regularity at* x*.*
- $g_x(X_i) = x_i$, $(\Pi X_i)(x) = x_i$. *(Canonical definition.)*
- $g(\tau)$ *and* $\Pi\sigma$ *rapidly decrease at* ∞ *for any non-polynomial* $\tau \in T^+$ *and* $\sigma \in T$*.*

Each element of $\mathcal{M}_{\mathrm{rap}}(\mathbb{R}^d)$ *is called a model.*

In general, $\mathcal{M}_{\mathrm{rap}}(\mathbb{R}^d)$ is not a linear space.

Controlled path and modelled distribution

We consider a "fibre bundle" on the Lie group G.

- A path $t \mapsto Y_t$ is said to be controlled by a path X if

$$Y_t = Y_s + Y'_s(X_t - X_s) + (\text{regular enough}).$$

- A T-valued field $f = \sum_{\tau \in \mathcal{B}} f_\tau \tau$ on $\mathbb{R}^d$ is said to be modelled by a model g if

$$f_\tau(y) = \sum_{|\sigma| \geq |\tau|} f_\sigma(x) g_{yx}(\sigma/\tau) + (\text{regular enough}),$$

where $g_{yx} := g_y * g_x^{-1}$.

Modelled distribution

Definition

Let $\gamma \in \mathbb{R}$ *and* $M = (g, \Pi) \in \mathcal{M}_{\mathrm{rap}}(\mathbb{R}^d)$. *The space* $\mathcal{D}^{\gamma}_{\mathrm{rap}}(g; \mathbb{R}^d)$ *consists of all* T*-valued functions*

$$f(x) = \sum_{\tau \in \mathcal{B}, |\tau| < \gamma} f_\tau(x)\tau, \quad x \in \mathbb{R}^d$$

such that

$$f_\tau(y) = \sum_{\sigma \in \mathcal{B}, |\sigma| \geq |\tau|} f_\sigma(x) g_{yx}(\sigma/\tau) + O(|y - x|^{\gamma - |\tau|}), \quad \tau \in \mathcal{B}.$$

In general, $\mathcal{D}^{\gamma}_{\mathrm{rap}}(g; \mathbb{R}^d)$ is not a direct product of Banach spaces.

Assumptions

Our algebra is compatible with the polynomial structure.

Assumption

The basis $\mathcal{B}^+$ of T^+ is a graded monoid generated by $\mathcal{B}_\circ^+$, that is the set consisting of

- *Taylor monomials $X_1, \dots, X_d$.*
- *Derivatives $\partial^k \tau$ of pure elements $\tau \in \mathcal{P}\mathcal{B}_\circ^+$.*

The coproduct $\Delta^+\tau$ is of the form

$$\Delta^+\tau = \sum_{\sigma \notin \{X^k\}} \sigma \otimes (\tau/\sigma) + \sum_{k \in \mathbb{N}^d} \frac{X^k}{k!} \otimes \partial^k \tau,$$

where $\partial^k \tau$ is defined for not pure τ by Leibniz rule.

BHZ Hopf algebra satisfies this assumption.

Assumptions

Assumption

The basis $\mathcal{B}$ of T is of the form $\mathcal{B} = \mathbb{N}^d \times \mathcal{B}_\circ$, i.e.

$$\mathcal{B} = \{X^k\tau;\, k \in \mathbb{N}^d, \tau \in \mathcal{B}_\circ\}.$$

For any $\tau \in \mathcal{B}_\circ$, its coproduct $\Delta\tau$ does not have components of the form

$$\sigma \otimes X^k$$

BHZ Hopf algebra does not seem to satisfy this assumption. Indeed,

$$\Delta I(X\Xi) = I(\Xi) \otimes X + \cdots .$$

However,

Proposition (Bailleul-H. (in preparation))

There is another basis $\tilde{\mathcal{B}}$ of BHZ algebra which satisfies the above assumption.

Theorem (Bailleul-H. (2018))

Let $M = (\Pi, g) \in \mathcal{M}_{\mathrm{rap}}(\mathbb{R}^d)$. There exist continuous linear maps

$$[\cdot]^M : T \to \mathcal{S}'(\mathbb{R}^d), \quad [\cdot]^g : T^+ \to \mathcal{C}(\mathbb{R}^d)$$

with the following properties.

- *For any $\tau \in T_\alpha$, one has $[\tau]^M \in \mathcal{C}^\alpha$, and*

$$\Pi\tau = \sum_{\eta \in \mathcal{B}, |\eta| < \alpha} g(\tau/\eta) \otimes [\eta]^M + [\tau]^M.$$

- *For any $\sigma \in T_\alpha^+$, one has $[\sigma]^g \in \mathcal{C}^\alpha$, and*

$$g(\sigma) = \sum_{\zeta \in \mathcal{B}^+, |\zeta| < \alpha} g(\sigma/\zeta) \otimes [\zeta]^g + [\sigma]^g.$$

In general, $\Pi\tau \notin C^{|\tau|}$ nor $g(\sigma) \notin C^{|\sigma|}$.

Theorem (Bailleul-H. (in preparation))

A subfamily

$$\{[\tau]^M; |\tau| < 0\}, \quad \{[\sigma]^g; \sigma \in \mathcal{PB}_\circ^+\}$$

is sufficient to *reassemble* *the model M. The inverse map is continuous. In other words, a* *homeomorphism*

$$\mathcal{M}_{\mathrm{rap}}(\mathbb{R}^d) \simeq \prod_{\tau \in \mathcal{B}_\circ,\, |\tau|<0} \mathcal{C}_{\mathrm{rap}}^{|\tau|}(\mathbb{R}^d) \times \prod_{\sigma \in \mathcal{PB}_\circ^+} \mathcal{C}_{\mathrm{rap}}^{|\sigma|}(\mathbb{R}^d)$$

is obtained.

Modelled distribution $\Rightarrow$ Paracontrolled distribution

Proposition (Bailleul-H. (2018))

Let $\gamma \in \mathbb{R}$ *and* $M = (g, \Pi) \in \mathcal{M}_{\mathrm{rap}}(\mathbb{R}^d)$. *For any modelled distribution* $f = \sum_{|\tau|<\gamma} f_\tau \tau \in \mathcal{D}^{\gamma}_{\mathrm{rap}}(g; \mathbb{R}^d)$, *one has*

$$f_\sigma = \sum_{|\sigma|<|\tau|<\gamma} f_\tau \otimes [\tau/\sigma]^g + [f/\sigma]^g, \quad \sigma \in \mathcal{B},$$

with $[f/\sigma]^g \in \mathcal{C}^{\gamma-|\sigma|}_{\mathrm{rap}}$. *Moreover, the reconstruction* $\mathcal{R}^M f$ *has the form*

$$\mathcal{R}^M f = \sum_{|\tau|<\gamma} f_\tau \otimes [\tau]^M + [f]^M,$$

where $[f]^M \in \mathcal{C}^{\gamma}_{\mathrm{rap}}$.

- $\mathcal{R}f$ is a reconstruction $\Leftrightarrow$ $\mathcal{R}f - \Pi_x f(x)$ has a regularity $\mathcal{C}^\gamma$ at x.
- $\mathcal{R}f$ is unique if $\gamma > 0$.

Modelled distribution $\Leftarrow$ Paracontrolled distribution

Theorem (Bailleul-H. (in preparation))

A subfamily

$$\{[f/\tau]^g; \tau \in \mathcal{B}_\circ\}$$

is sufficient to reassemble the modelled distribution f. The inverse map is continuous. In other words, a homeomorphism

$$\mathcal{D}^\gamma_{\mathrm{rap}}(g;\mathbb{R}^d) \simeq \prod_{\tau\in\mathcal{B}_\circ, |\tau|<\gamma} \mathcal{C}^{\gamma-|\tau|}_{\mathrm{rap}}(\mathbb{R}^d)$$

is obtained.

Outline of the proof

- $\sum_{i\geq -1}\rho_i = 1$: dyadic partition of unity.
- $K_i = \hat{\rho}_i$, $K_{<i} := \sum_{j\leq i-2} K_j$.
- We define $P : \mathcal{C}(\mathbb{R}^d \times \mathbb{R}^d) \to \mathcal{C}(\mathbb{R}^d)$ by

$$PF(x) := \sum_i \iint K_{<i}(x-y)K_i(x-z)F(y,z)dydz.$$

If $F(y,z) = f(y)g(z)$, then

$$PF = f \otimes g.$$

Lemma

Let $\alpha > 0$. If $|F(y,z)| \lesssim |y-z|^\alpha$, then $PF \in \mathcal{C}^\alpha$.

We consider (T^+, Δ^+). By the formula $\Delta^+\tau = \sum \sigma \otimes (\tau/\sigma)$, we have

$$g_z(\tau) = \sum g_{zy}(\sigma) g_y(\tau/\sigma).$$

By assumption that $\Delta^+\tau = \tau \otimes \mathbf{1} + \mathbf{1} \otimes \tau + \sum_{0<|\sigma|<|\tau|} \sigma \otimes (\tau/\sigma)$,

$$g_{zy}(\tau) = g_z(\tau) - g_y(\tau) - \sum_{0<|\sigma|<|\tau|} g_y(\tau/\sigma) g_{zy}(\sigma).$$

Repeating this expansion,

$$\begin{aligned} g_{zy}(\tau) &= g_z(\tau) - g_y(\tau) \\ &\quad - \sum_{n=1}^{\infty} (-1)^{n-1} \sum_{0<|\sigma_n|<\cdots<|\sigma_1|<|\tau|} (g_{\sigma_1}^{\tau} \cdots g_{\sigma_n}^{\sigma_{n-1}})(y)(g_z(\sigma_n) - g_y(\sigma_n)), \end{aligned}$$

where $g_\sigma^\tau(z) := g_z(\tau/\sigma)$.

Applying the operator P,

$$(\mathcal{C}^{|\tau|}) = g(\tau) - \sum_{n=1}^{\infty}(-1)^{n-1} \sum_{0<|\sigma_n|<\cdots<|\sigma_1|<|\tau|} (g_{\sigma_1}^{\tau} \cdots g_{\sigma_n}^{\sigma_{n-1}}) \otimes g(\sigma_n).$$

By induction that $g(\sigma_n) = \sum_{0<|\sigma_{n+1}|<|\sigma_n|} g_{\sigma_{n+1}}^{\sigma_n} \otimes [\sigma_{n+1}]^g + [\sigma_n]^g$, we have

$$g(\tau) = (\mathcal{C}^{|\tau|}) + \sum_{0<|\sigma|<|\tau|} g_{\sigma}^{\tau} \otimes [\sigma]^g$$

$$+ \sum_{n=1}^{\infty}(-1)^{n-1} \sum_{0<|\sigma_{n+1}|<\cdots<|\sigma_1|<|\tau|} R(g_{\sigma_1}^{\tau} \cdots g_{\sigma_n}^{\sigma_{n-1}}, g_{\sigma_{n+1}}^{\sigma_n}, [\sigma_{n+1}]^g),$$

where

$$R(f, g, h) := f \otimes (g \otimes h) - (fg) \otimes h.$$

By using the expansion of $g_{zy}(\tau)$ again we can show that $\sum R \in \mathcal{C}^{|\tau|}$.

Future works

- Group actions and renormalizations done in the simple space $\prod \mathcal{C}$.
  - Pararemainders $[\tau]^M$ for $|\tau| < 0$ are exactly what should be renormalized.
  - Tapia-Zambotti (2018): Free and transitive action of a space $\prod_\tau \mathcal{C}^{\alpha|\tau|}$ on the space of branched RPs.
- Systematic approach to PC.
  - There were no theories to give a structural meaning to pararemainders.
- Application to real analysis.
  - Hoshino (2019): A systematic proof of the "commutator estimate".