PARACONTROLLED CALCULUS AND REGULARITY STRUCTURES

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Regularity structure (RS) by Hairer (2014) and *Paracontrolled calculus* (PC) by Gubinelli-Imkeller-Perkowski (2015) both solve many singular SPDEs. These theories are believed to be equivalent, but there are some gaps. For example, the general KPZ equation

$$\partial_t h = \partial_x^2 h + f(h)(\partial_x h)^2 + g(h)\xi$$

is solved by RS, but cannot be solved within PC. Our aim is to show the equivalence between RS and (an extension of) PC and fill such gaps.

Both of RS and PC are extensions of the *rough path theory*. Our main result means that the two different ways of defining the "rough paths" on the same algebraic structure are equivalent. A rough image of our main result is the following.

Theorem 1. [1, 2] *The following equivalences hold.*

Rough path theory	*RS*		*PC*
Rough path	*Model*	$\Leftrightarrow$ [1]	*Pararemainders*
Controlled path	*Modelled distribution*	$\Leftrightarrow$ [2]	*Paracontrolled distribution*

We explain the precise meanings. Recall that the α-Hölder geometric rough paths living in $\mathbb{R}^n$ are α-Hölder continuous paths on the $\lfloor 1/\alpha \rfloor$-step nilpotent Lie group $G^{(\lfloor 1/\alpha \rfloor)}(\mathbb{R}^n)$, which is identified with a space of linear functionals g on $T^{(\lfloor 1/\alpha \rfloor)}(\mathbb{R}^n)$ such that

$$g(x \sqcup\!\sqcup y) = g(x)g(y)$$

for any $x, y \in T^{(\lfloor 1/\alpha \rfloor)}(\mathbb{R}^n)$, where $\sqcup\!\sqcup$ is the shuffle product. The model is defined similarly by replacing $T^{(\lfloor 1/\alpha \rfloor)}(\mathbb{R}^n)$ by a general graded Hopf algebra T^+.

Definition 1. *A* (concrete) regularity structure *is a pair of a graded Hopf algebra* $T^+ = \bigoplus_{\alpha \in A^+} T_\alpha^+$ *with a coproduct* Δ^+ *and a graded linear space* $T = \bigoplus_{\beta \in A} T_\beta$ *with a linear map* $\Delta : T \to T \otimes T^+$ *satisfying the right comodule properties on* T^+ *such that*

$$\Delta^+ \tau \in \tau \otimes \mathbf{1} + \mathbf{1} \otimes \tau + \bigoplus_{0<\beta<\alpha} T_\beta^+ \otimes T_{\alpha-\beta}^+,$$

$$\Delta \sigma \in \sigma \otimes \mathbf{1} + \bigoplus_{\gamma<\beta} T_\gamma \otimes T_{\beta-\gamma}^+,$$

for any $\tau \in T_\alpha^+$ *and* $\sigma \in T_\beta$.

Let $\mathcal{B}_\alpha^{(+)}$ be a basis of $T_\alpha^{(+)}$ and let $\mathcal{B}^{(+)} = \bigcup_{\alpha \in A^{(+)}} \mathcal{B}_\alpha^{(+)}$. We assume that the basis $\mathcal{B}^+$ is a monoid generated by $\mathcal{B}_\circ^+$. Moreover, we assume that $\mathcal{B}_\circ^+$ consists of

- monomials $X_1, \dots, X_d$,
- derivatives $\partial^k \tau$ of "pure" elements $\tau \in \mathcal{P}\mathcal{B}_\circ^+$.

Then we have the following equivalence result. We say that each element $\tau \in T_\alpha^{(+)}$ is homogeneous and write

$$|\tau| = \alpha.$$

Theorem 2. [1, 2] *Let $\mathcal{M}_{\mathrm{rap}}$ be the space of all* models (g, Π) *on $\mathbb{R}^d$, i.e., all pairs of*

- *a Lipschitz continuous map g from $\mathbb{R}^d$ to the group G of algebra homomorphisms $T^+ \to \mathbb{R}$,*
- *a bounded operator $\Pi : T \to \mathcal{S}'(\mathbb{R}^d)$ such that $\Pi\tau$ has a "$|\tau|$-class Taylor expansion" for any homogeneous $\tau \in T$,*

and such that $\mathsf{g}(\tau)$ and $\Pi\sigma$ rapidly decrease at infinity for any $\tau \in T^+$ and $\sigma \in T$. Then the space $\mathcal{M}_{\mathrm{rap}}$ is homeomorphic to the direct product of Banach spaces;

$$\mathcal{M}_{\mathrm{rap}} \simeq \prod_{\tau \in \mathcal{B}_\circ,\, |\tau|<0} \mathcal{C}_{\mathrm{rap}}^{|\tau|}(\mathbb{R}^d) \times \prod_{\sigma \in \mathcal{PB}_\circ^+} \mathcal{C}_{\mathrm{rap}}^{|\sigma|}(\mathbb{R}^d).$$

We return to the rough path theory. The path $Y : [0,T] \to \mathbb{R}$ is said to be an α-Hölder path controlled by a rough path $\mathbf{X}$ if there exists a continuous path $\mathbf{Y} : [0,T] \to T^{(\lfloor 1/\alpha \rfloor -1)}(\mathbb{R}^n)$ such that $Y_t^{\emptyset} = Y_t$ and

$$Y_t^{i_1\dots i_k} = \sum_{i_{k+1},\dots,i_\ell} Y_s^{i_1\dots i_k i_{k+1}\dots i_\ell} \mathbf{X}_{st}^{i_1\dots i_k i_{k+1}\dots i_\ell} + O(|t-s|^{(\lfloor 1/\alpha \rfloor - k)\alpha})$$

for any $i_1, \dots, i_k \in \{1, \dots, n\}$, where $Y^{i_1\dots i_k}$ and $\mathbf{X}^{i_1\dots i_k}$ represents the $e_{i_1} \otimes \cdots \otimes e_{i_k}$-components of Y and $\mathbf{X}$, respectively. A modelled distribution is a T-valued function on $\mathbb{R}^d$ with a similar "Taylor-like expansion" at each point $x \in \mathbb{R}^d$. Note that any $\mathsf{g} \in G$ acts on T by

$$\hat{\mathsf{g}}(\tau) := (\mathrm{Id} \otimes \mathsf{g})\Delta\tau, \quad \tau \in T.$$

Denote by $(\cdot)_\tau : T \to \mathbb{R}$ the projection to the τ-component.

Theorem 3. [2] *Let $\gamma \in \mathbb{R}$. Let $\mathcal{D}_{\mathrm{rap}}^\gamma$ be the space of all* γ-class modelled distributions*, i.e., all functions $f : \mathbb{R}^d \to T$ such that, for any homogeneous $\tau \in T$,*

$$\big(f(y) - \hat{\mathsf{g}}_{yx} f(x)\big)_\tau = O(|y-x|^{\gamma - |\tau|}),$$

where $\mathsf{g}_{yx} := \mathsf{g}_y \mathsf{g}_x^{-1} \in G$, and $\big(f(x)\big)_\tau$ rapidly decreases as $|x| \to \infty$.

We assume that the basis $\mathcal{B}$ of T has a good structure; monomials and antiderivatives. Then the space $\mathcal{D}_{\mathrm{rap}}^\gamma$ is homeomorphic to the direct product of Banach spaces;

$$\mathcal{D}_{\mathrm{rap}}^\gamma \simeq \prod_{\tau \in \mathcal{PB}_\circ,\, |\tau|<\gamma} \Big(\prod_{\eta \in \mathcal{PB}_\tau} \mathcal{C}_{\mathrm{rap}}^{\gamma - |\eta|}(\mathbb{R}^d) \Big),$$

where $\mathcal{PB}_\circ$ is the set of all "pure" elements, and $\mathcal{PB}_\tau$ is the set of independent antiderivatives of τ. (If the antiderivative is unique for each τ, then $\mathcal{PB}_\tau = \{\tau\}$.)

This talk is based on a joint work with Ismaël Bailleul (Université de Rennes 1).

References

[1] I. Bailleul and M. Hoshino, *Paracontrolled calculus and regularity structures (1)*, arXiv:1812.07919.
[2] I. Bailleul and M. Hoshino, *Paracontrolled calculus and regularity structures (2)*, in preparation.

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