A limit theorem for inverse local times of jumping-in diffusion processes

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We study a strong Markov process X on the half line $[0, \infty)$ which is a natunal scale diffusion up to the first hitting time of 0 and, as soon as X hits 0, X jumps into the interior $(0, \infty)$ and starts afresh. It was shown by Feller[1] and Itô[2] that such a process can be characterized by the speed measure dm which characterizes the diffusion on the interior $(0, \infty)$ and the jumping-in measure j which characterizes the law of jumps from the boundary 0 to the interior $(0, \infty)$. We denote this process by $X_{m,j}$ and call it a *jumping-in diffusion*.

Let us consider the inverse local time $\eta_{m,j}$ at 0 of a jumping-in diffusion $X_{m,j}$. One of our two main theorems is to establish the fluctuation scaling limit of the inverse local time $\eta_{m,j}$ of the form:

$$\frac{1}{\lambda^{1/\alpha}}(\eta_{m,j}(\lambda t) - b\lambda t) \xrightarrow[\lambda\to\infty]{d} T(t) \text{ in } \mathbb{D} \tag{1}$$

for some constants $b \geq 0$ and $\alpha \in (0, 2]$. Here $\mathbb{D}$ denotes the space of càdlàg paths from $[0, \infty)$ to $\mathbb{R}$ equipped with Skorokhod's J_1-topology. In order to obtain the limit, we establish the continuity theorem for jumping-in diffusion processes which roughly asserts the following: for jumping-in diffusions $\{X_{m_n,j_n}\}_n$, if their speed measures $\{dm_n\}_n$ converge to a speed measure dm in a certain sense and the measures $\{j_n(dx)\}_n$ degenerate to the point mass at the origin in a certain sense, then for the appropriate constants $\{b_n\}_n$, it holds that

$$\eta_{m_n,j_n}(t) - b_n t \xrightarrow[n\to\infty]{d} \sigma B(t) + T(m; \kappa t) \text{ in } \mathbb{D} \tag{2}$$

for some constants σ and κ. Here B denotes a standard Brownian motion and $T(m; t)$ the spectrally positive Lévy process associated to m which is independent of B. In order to prove the continuity theorem, we introduce a class of λ-eigenfunctions of the generalized second order differential operator $\frac{d}{dm}\frac{d}{dx}$ and apply Krein-Kotani correspondence and its continuity established in Kotani[4].

The other one is about the occupation time of two-sided jumping-in diffusions which are constructed by connecting two jumping-in diffusion processes with respect to 0. Let X be such a process and define $A(t) = \int_0^t 1_{(0,\infty)}(X_s)ds$. We give conditions for the existence of the limit distribution $\frac{1}{t}A(t)$ as $t \to \infty$. Moreover, in the case where the limit degenerate, that is,

$$\frac{1}{t}A(t) \xrightarrow[t\to\infty]{P} p \in [0,1] \tag{3}$$

holds, we show the scaling limit of the fluctuation around the limit constant along the exponential clock, that is, the following limit:

$$\frac{1}{\mathbf{e}_{f(q)}}(A(\mathbf{e}_q t) - p\mathbf{e}_q t) \xrightarrow[q\to+0]{d} Z(t) \tag{4}$$

for some positive function $f(q)$ which diverges to ∞ as $q \to +0$. Here $\mathbf{e}_q$ denotes an exponentially distributed random variable with parameter $q > 0$ and is independent of X. This result is a jumping-in version of the result proved for diffusions in Kasahara and Watanabe[3].

References

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