On the pathwise uniqueness of solutions of SDEs driven by Lévy processes

Hiroshi TSUKADA (Kyoto University)

joint work with Atsushi TAKEUCHI (Osaka City University)

November 20, 2018

Outline

SDEs driven by Lévy processes

- Let $W = \{W_t : t \geq 0\}$ be a 1-dimensional Brownian motion.
- Let $Z = \{Z_t : t \geq 0\}$ be a 1-dimensional Lévy process.

Consider 1-dimensional stochastic differential equations (SDEs):

$$X_t = x + \int_0^t a(X_s)\, ds + \int_0^t b(X_s)\, dW_s + \int_0^t c(X_{s-})\, dZ_s.$$

In this talk, we shall study the pathwise uniqueness (PU) of the solutions to the SDE.

Weak solutions of the SDE

Throughout this talk, we assume the following:

Assumption

Let a, b, $c : \mathbb{R} \to \mathbb{R}$ be continuous with the linear growth condition.

- Under this conditon, the SDE has a weak solution. (Situ's book (2005))

In this talk, we shall study the condition on the coefficients a, b, c under which the PU of solutions of the SDEs can be justified. Our approach is based on Gronwall's inequality.

SDEs driven by BM

Consider the SDE driven by BM $(a = 0, c = 0)$:

$$X_t = x + \int_0^t b(X_s)\, dW_s.$$

- PU holds if b is locally $1/2$-Hölder continuous. (Yamada and Watanabe (1971))

Remark

There are counter exapmles if b is δ-Hölder continuous for $0 < \delta < 1/2$.

SDEs driven by stable processes

When Z is a strictly stable process of index $\alpha \in (1, 2)$ with parameter (r_-, r_+), consider the SDE driven by stable processes $(a = 0, b = 0)$:

$$X_t = x + \int_0^t c(X_{s-})\, dZ_s.$$

- Z: symmetric stable process $(r_- = r_+)$.
  PU holds if c is $1/\alpha$-Hölder continuous. (Komatsu (1982))
- Z: spectrally positive stable process $(r_- = 0)$.
  PU holds if c is increasing and $(\alpha - 1)/\alpha$-Hölder continuous. (Li and Mytnik (2011))
- Z: stable process $(r_+ \geq r_-)$.
  There exists $\beta = \beta(\alpha, r_-/r_+)$ such that the PU holds when c is increasing and $(\alpha - \beta)/\alpha$-Hölder continuous. (Fournier (2013))
  Remark that $(\alpha - \beta)/\alpha \in [(\alpha - 1)/\alpha, 1/\alpha]$.

Our class of driving processes Z

In this talk, we shall study the problem on the PU in the case of the Lévy process with the triplet $\left(\gamma_\rho^{\alpha_-,\,\alpha_+},\, 0,\, \nu_\rho^{\alpha_-,\,\alpha_+}\right)$ given by

$$\nu_\rho^{\alpha_-,\,\alpha_+}(dz) = \rho(z)\left(|z|^{-\alpha_- -1}\,\mathbb{I}_{(z<0)} + |z|^{-\alpha_+ -1}\,\mathbb{I}_{(z>0)}\right)\,dz,$$
$$\gamma_\rho^{\alpha_-,\,\alpha_+} = -\int_{|z|>1} z\,\nu_\rho^{\alpha_-,\alpha_+}(dz),$$

where $\rho : \mathbb{R}_0 \to [0,+\infty)$ is a bounded measurable function such that

$$\rho(0+) = \lim_{z\to 0+} \rho(z) > 0, \quad \rho(0-) = \lim_{z\to 0-} \rho(z) \geq 0,$$

and $\alpha_-,\, \alpha_+ \in (1,\, 2)$ such that $\alpha_- \leq \alpha_+$.
(Remark that $\mathbb{R}_0 := \mathbb{R}\setminus\{0\}$.)

Examples: our driving process Z

Example 1 (Stable processes)

The Lévy measure of a stable process is

$$\nu(dz) = |z|^{-\alpha-1} \left(r_- \, \mathbb{I}_{(z<0)} + r_+ \, \mathbb{I}_{(z>0)}\right) dz,$$

where $\alpha \in (1,\, 2)$ and $r_-,\, r_+$ are constants such that $0 \leq r_- \leq r_+$.

Example 2 (Truncated stable processes)

The Lévy measure of a truncated stable process is

$$\nu(dz) = |z|^{-\alpha-1} \left(r_- \, \mathbb{I}_{(-1<z<0)} + r_+ \, \mathbb{I}_{(0<z<1)}\right) \, dz,$$

where $\alpha \in (1,\, 2)$ and $r_-,\, r_+$ are constants such that $0 \leq r_- \leq r_+$.

Example 3 (Tempered stable processes)

The Lévy measure of a tempered stable process is

$$\nu(dz) = \left(r_- \, |z|^{-\sigma_- -1} \, e^{-\lambda_- \, |z|} \, \mathbb{I}_{(z<0)} + r_+ \, |z|^{-\sigma_+ -1} \, e^{-\lambda_+ \, |z|} \, \mathbb{I}_{(z>0)} \right) \, dz,$$

where $\sigma_- \in (0,\, 2)$, $\sigma_+ \in (1,\, 2)$ and $r_-,\, r_+,\, \lambda_-,\, \lambda_+$ are constants such that $r_-,\, r_+,\, \lambda_-,\, \lambda_+ > 0$.

Example 4 (Relativistic stable processes)

The Lévy measure of a relativistic stable process is

$$\nu(dz) = r \, |z|^{-\alpha-1} \left(\int_0^\infty s^{(1+\alpha)/2-1} \, \exp\left(-\frac{s}{4} - \frac{m^{2/\alpha} \, |z|^2}{s} \right) \, ds \right) dz,$$

where $\alpha \in (1,\, 2)$ and $m,\, r$ are constants such that $m,\, r > 0$.

Estimate on α-stable Lévy measure

Let $\alpha \in (1,2)$, $0 \le r_- \le r_+$, $0 < \beta \le 1$, and introduce

$$I^{\alpha,\,\beta}_{r_-,\,r_+} = \int_{\mathbb{R}_0} \left\{ |1+z|^{\beta} - 1 - \beta\, z \right\} \nu^{\alpha}_{r_-,r_+}(dz),$$

where $\nu^{\alpha}_{r_-,r_+}$ is the stable Lévy measure given by

$$\nu^{\alpha}_{r_-,r_+}(dz) = |z|^{-\alpha-1} \left(r_- \, \mathbb{I}_{(z<0)} + r_+ \, \mathbb{I}_{(z>0)} \right) dz.$$

Lemma 1

For $u \in [0,\,1]$, *define*

$$\beta(\alpha,\,u) := \frac{1}{\pi} \arccos\left[\frac{u^2 \sin^2(\pi\,\alpha) - (1 + u\cos(\pi\,\alpha))^2}{u^2 \sin^2(\pi\,\alpha) + \left(1 + u\cos(\pi\,\alpha)\right)^2} \right] \in [\alpha - 1,\, 1].$$

Set $\beta_0 = \beta(\alpha,\, r_-/r_+)$. *Then it holds* $I^{\alpha,\,\beta_0}_{r_-,r_+} = 0$. *Furthermore, it holds that* $I^{\alpha,\,\beta}_{r_-,r_+} < 0$ *for* $\beta \in (0,\, \beta_0)$.

Proof of Lemma 1

By using integration by parts, we have

$$I^{\alpha,\beta}_{r_-,r_+} = \frac{\beta\,\Gamma(\beta)\,\Gamma(\alpha-\beta)}{\alpha\,\Gamma(\alpha)\,\sin\big(\pi(\alpha-1)\big)} \times \left\{-r_+\,\sin(\pi\beta) + r_-\,\sin\big(\pi(\alpha-1)\big) + r_-\,\sin\big(\pi(\alpha-\beta)\big)\right\}.$$

By setting $u = r_-/r_+$, it is enough to show for $\beta \in (0,\,\beta_0)$,

$$-\sin(\pi\,\beta) + u\,\sin\big(\pi(\alpha-1)\big) + u\,\sin\big(\pi(\alpha-\beta)\big) < 0$$
$$\iff\quad u^2\,\big(1-\cos(\pi\,\beta)\big)\,\sin^2(\pi\,\alpha) < \big(1+u\,\cos(\pi\,\alpha)\big)^2\,\big(1+\cos(\pi\,\beta)\big).$$

The inequality follows from

$$\cos(\pi\,\beta) > \cos(\pi\,\beta_0) = \frac{u^2\,\sin^2(\pi\,\alpha) - \big(1+u\,\cos(\pi\,\alpha)\big)^2}{u^2\,\sin^2(\pi\,\alpha) + \big(1+u\,\cos(\pi\,\alpha)\big)^2}. \quad \square$$

Main result ($\alpha_- = \alpha_+$)

Theorem 1 (Case of $\alpha_- = \alpha_+$)

Let $\alpha_- = \alpha_+ =: \alpha$ *and* $\rho(0-) \le \rho(0+)$. *Write* $\beta_0 = \beta(\alpha, \rho(0-)/\rho(0+))$ *as introduced in Lemma 1. Suppose that the coefficients* a, b, c *satisfy*

(i) a *is decreasing;*

(ii) b *is locally* $(2-\beta)/2$*-Hölder continuous with* $\beta \in (0, \beta_0)$*;*

(iii) c *is increasing and locally* $(\alpha-\beta)/\alpha$*-Hölder continuous with* $\beta \in (0, \beta_0)$.

Then, the PU holds.

Main result ($\alpha_- < \alpha_+$)

Theorem 2 (Case of $\alpha_- < \alpha_+$)

Let $\alpha_- < \alpha_+$. Suppose that the coefficients a, b, c satisfy

(i) *a is decreasing;*

(ii) *b is locally $(2 - \beta)/2$-Hölder continuous with $\beta \in (0, 1)$;*

(iii) *c is increasing and locally $(\alpha_+ - \beta)/\alpha_+$-Hölder continuous with $\beta \in (0, 1)$.*

Then, the PU holds.

Driving process Z

By the Lévy-Itô decomposition,

$$Z_t = \int_0^t \int_{\mathbb{R}_0} z \, \tilde{N}(dz, ds),$$

where $N(dz, ds)$ is the Poisson random measure with the intensity $\hat{N}(dz, ds) := \nu_\rho^{\alpha_-, \alpha_+}(dz) \, ds$, and the compensated random measure $\tilde{N}(dz, ds) = N(dz, ds) - \hat{N}(dz, ds)$.

Consider 1-dimensional SDEs:

$$\begin{aligned} X_t &= x + \int_0^t a(X_s) \, ds + \int_0^t b(X_s) \, dW_s + \int_0^t c(X_{s-}) \, dZ_s \\ &= x + \int_0^t a(X_s) \, ds + \int_0^t b(X_s) \, dW_s + \int_0^t \int_{\mathbb{R}_0} c(X_{s-}) \, z \, \tilde{N}(dz, ds). \end{aligned}$$

Notations

For $i = 1,\, 2$, we consider two solutions of the SDE:

$$X_t^i = x + \int_0^t a(X_s^i)\, ds + \int_0^t b(X_s^i)\, dW_s + \int_0^t \int_{\mathbb{R}_0} c(X_{s-}^i)\, z\, \tilde{N}(dz, ds).$$

Write $\Delta_t = X_t^1 - X_t^2$ and

$$A_t = a(X_t^1) - a(X_t^2), \quad B_t = b(X_t^1) - b(X_t^2), \quad C_t = c(X_t^1) - c(X_t^2).$$

Then, we are in position to study

$$\Delta_t = \int_0^t A_s\, ds + \int_0^t B_s\, dW_s + \int_0^t \int_{\mathbb{R}_0} C_{s-}\, z\, \tilde{N}(dz, ds).$$

Itô formula for Φ_η

Let $0 < \beta \leq 1$, $0 < \eta \leq 1$ and $N > 0$.
Define the function Φ_η and the stopping time T_N by

$$\Phi_\eta(u) = \left(u^2 + \eta^2\right)^{\beta/2}, \quad T_N = \inf\left\{t > 0\,;\, |X_t^1| \wedge |X_t^2| > N\right\}.$$

Now, we shall apply the Itô formula for the function Φ_η and get

$$\begin{aligned}
&\Phi_\eta(\Delta_{t\wedge T_N}) - \Phi_\eta(0) \\
&= \int_0^{t\wedge T_N} \left\{\Phi_\eta'(\Delta_s)\, A_s + \frac{1}{2}\, \Phi_\eta''(\Delta_s)\, B_s^2\right\} ds + \int_0^{t\wedge T_N} \Phi_\eta'(\Delta_s)\, B_s\, dW_s \\
&\quad + \int_0^{t\wedge T_N}\int_{\mathbb{R}_0} \left\{\Phi_\eta(\Delta_{s-} + C_{s-}\, z) - \Phi_\eta(\Delta_{s-})\right\} \tilde{N}(dz, ds) \\
&\quad + \int_0^{t\wedge T_N}\int_{\mathbb{R}_0} \left\{\Phi_\eta(\Delta_s + C_s\, z) - \Phi_\eta(\Delta_s) - \Phi_\eta'(\Delta_s)\, C_s\, z\right\} \hat{N}(dz, ds).
\end{aligned}$$

Expectation of the Itô formula for Φ_η

Taking the expectation implies that

$$\begin{aligned}
&\mathbb{E}\big[\Phi_\eta(\Delta_{t\wedge T_N})\big] - \Phi_\eta(0) \\
&= \mathbb{E}\left[\int_0^{t\wedge T_N} \Phi_\eta'(\Delta_s)\, A_s\, ds\right] && \text{(A)} \\
&\quad + \mathbb{E}\left[\int_0^{t\wedge T_N} \frac{1}{2}\, \Phi_\eta''(\Delta_s)\, B_s^2\, ds\right] && \text{(B)} \\
&\quad + \mathbb{E}\left[\int_0^{t\wedge T_N}\int_{\mathbb{R}_0} \big\{\Phi_\eta(\Delta_s + C_s\, z) - \Phi_\eta(\Delta_s) - \Phi_\eta'(\Delta_s)\, C_s\, z\big\}\ \hat{N}(dz, ds)\right]. \\
& && \text{(C)}
\end{aligned}$$

The monotone convergence theorem yields that

$$\mathbb{E}\big[\Phi_\eta(\Delta_{t\wedge T_N})\big] \to \mathbb{E}\big[|\Delta_{t\wedge T_N}|^\beta\big]$$

as $\eta \searrow 0$. Thus, we shall study the limit of the right hand side

Limit of (A) as $\eta \searrow 0$

Lemma 2

Suppose that a is monotonic. Then, it holds that

$$\lim_{\eta \searrow 0} \mathbb{E}\left[\int_0^{t\wedge T_N} \Phi'_\eta(\Delta_s)\, A_s\, ds\right] = \mathbb{E}\left[\int_0^{t\wedge T_N} \beta\, \mathit{sgn}(\Delta_s)\, |\Delta_s|^{\beta-1} A_s\, ds\right].$$

Proof. The monotone convergence theorem leads us to see that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{t\wedge T_N} \Phi'_\eta(\Delta_s)\, A_s\, ds\right] \\
&= \mathbb{E}\left[\int_0^{t\wedge T_N} \beta\, \Delta_s \left|\Delta_s^2 + \eta^2\right|^{(\beta-2)/2} A_s \left(\mathbb{I}_{(A_s\, \Delta_s \geq 0)} + \mathbb{I}_{(A_s\, \Delta_s < 0)}\right) ds\right] \\
&\to \mathbb{E}\left[\int_0^{t\wedge T_N} \beta\, \mathrm{sgn}(\Delta_s)\, |\Delta_s|^{\beta-1} A_s\, ds\right],
\end{aligned}$$

as $\eta \searrow 0$. □

Limit of (B) as $\eta \searrow 0$

Lemma 3

Suppose that, for each $N > 0$, there exists a positive constant $K_1(N)$ s.t.

$$|b(x) - b(\tilde{x})| \leq K_1(N)\,|x - \tilde{x}|^{(2-\beta)/2}, \tag{H1}$$

for all $|x|$, $|\tilde{x}| \leq N$. Then, it holds that

$$\lim_{\eta \searrow 0} \mathbb{E}\left[\int_0^{t \wedge T_N} \Phi_\eta''(\Delta_s)\,B_s^2\,ds\right] = \mathbb{E}\left[\int_0^{t \wedge T_N} \beta\,(\beta - 1)\,|\Delta_s|^{\beta-2}\,B_s^2\,ds\right].$$

Proof. Since $\Phi_\eta''(u) = \beta\,|u^2 + \eta^2|^{(\beta-2)/2} + \beta\,(\beta - 2)\,u^2\,|u^2 + \eta^2|^{(\beta-4)/2}$,

$$\left|\Phi_\eta''(\Delta_s)\,B_s^2\right| \leq \beta\,(3 - \beta)\,|\Delta_s|^{\beta-2}\,|B_s|^2 \leq K_1(N)^2\,\beta\,(3 - \beta).$$

Hence, the assertion follows from the dominated convergence theorem. □

Limit of (C) as $\eta \searrow 0$

Lemma 4

Suppose that, for each $N > 0$, there exists a positive constant $K_2(N)$ s.t.

$$|c(x) - c(\tilde{x})| \leq K_2(N)\,|x - \tilde{x}|^{(\alpha_+ - \beta)/\alpha_+}, \tag{H2}$$

for all $|x|$, $|\tilde{x}| \leq N$. Then, it holds that

$$\lim_{\eta \searrow 0} \mathbb{E}\left[\int_0^{t\wedge T_N}\int_{\mathbb{R}_0} \left\{\Phi_\eta(\Delta_s + C_s) - \Phi_\eta(\Delta_s) - \Phi'_\eta(\Delta_s)\,C_s\,z\right\}\nu_\rho^{\alpha_-,\alpha_+}(dz)ds\right]$$
$$= \mathbb{E}\left[\int_0^{t\wedge T_N} |\Delta_s|^\beta \left(\int_{\mathbb{R}_0}\left\{\left|1 + \frac{C_s}{\Delta_s}\,z\right|^\beta - 1 - \beta\,\frac{C_s}{\Delta_s}\,z\right\}\nu_\rho^{\alpha_-,\alpha_+}(dz)\right)ds\right].$$

Remark

$$(\alpha_+ - \beta)/\alpha_+ \geq (\alpha_- - \beta)/\alpha_-.$$

Proof of Lemma 4

Lemma 5

$$\left|\Phi_\eta(\Delta_s + C_s\, z) - \Phi_\eta(\Delta_s) - \Phi'_\eta(\Delta_s)\, C_s\, z\right|$$
$$\leq K_3(\beta)\left\{\left(|\Delta_s|^{\beta-2}\,|C_s\, z|^2\right) \wedge \left(|\Delta_s|^{\beta-1}\,|C_s\, z|\right)\right\}$$

on the event $\{\Delta_s \neq 0\}$, *where* $K_3(\beta)$ *is a positive constant.*

Lemma 6

Under the condition (H2), *it holds that*

$$\mathbb{E}\left[\int_0^{t\wedge T_N}\int_{\mathbb{R}_0}\left\{\left(|\Delta_s|^{\beta-2}\,|C_s\, z|^2\right) \wedge \left(|\Delta_s|^{\beta-1}\,|C_s\, z|\right)\right\}\nu_\rho^{\alpha_-,\,\alpha_+}(dz)\, ds\right] < \infty.$$

Itô formula for $|x|^\beta$

Corollary 1

Suppose that a is monotonic and the conditions (H1) *and* (H2) *hold. Then, it holds that*

$$
\begin{aligned}
&\mathbb{E}\left[|\Delta_{t\wedge T_N}|^\beta\right] \\
&= \mathbb{E}\left[\int_0^{t\wedge T_N} \beta\, \mathit{sgn}(\Delta_s)\, |\Delta_s|^{\beta-1} A_s\, ds\right] \\
&+ \mathbb{E}\left[\int_0^{t\wedge T_N} \frac{\beta}{2}(\beta-1)\, |\Delta_s|^{\beta-2} B_s^2\, ds\right] \\
&+ \mathbb{E}\left[\int_0^{t\wedge T_N} |\Delta_s|^\beta \left(\int_{\mathbb{R}_0} \left\{\left|1+\frac{C_s}{\Delta_s}z\right|^\beta - 1 - \beta\,\frac{C_s}{\Delta_s}\,z\right\} \nu_\rho^{\alpha_-,\alpha_+}(dz)\right) ds\right].
\end{aligned}
$$

Proof. Direct consequences of Lemmas 2, 3 and 4. □

Case of $\alpha_- = \alpha_+$

First, we shall prove the case of $\alpha_- = \alpha_+$.

For simplicity, write $\alpha := \alpha_- = \alpha_+$ and $\nu_\rho^\alpha := \nu_\rho^{\alpha_-,\alpha_+}$.

Lemma 7

Let $\beta_0 = \beta(\alpha,\, \rho(0-)/\rho(0+))$ as introduced in Lemma 1, and suppose that $\rho(0-) \le \rho(0+)$. Then, for each $\beta \in (0,\, \beta_0)$, it holds that

$$K_4(\beta) := \sup_{k \ge 0} \int_{\mathbb{R}_0} \left\{ |1 + k\, z|^\beta - 1 - \beta\, k\, z \right\} \nu_\rho^\alpha(dz) < +\infty.$$

Proof. From Lemma 1, we have

$$\int_{\mathbb{R}_0} \left\{ |1 + k\, z|^\beta - 1 - \beta\, k\, z \right\} \rho(z)\, |z|^{-\alpha-1}\, dz$$
$$= k^\alpha \int_{\mathbb{R}_0} \left\{ |1 + y|^\beta - 1 - \beta\, y \right\} \rho\left(\frac{y}{k}\right) |y|^{-\alpha-1}\, dy \to -\infty \quad \text{as } k \to +\infty.$$

Proof of our result ($\alpha_- = \alpha_+$)

Theorem 1 (Case of $\alpha_- = \alpha_+$)

Let $\alpha_- = \alpha_+ =: \alpha$ and $\rho(0-) \le \rho(0+)$. Write $\beta_0 = \beta(\alpha, \rho(0-)/\rho(0+))$ as introduced in Lemma 1. Suppose that the coefficients a, b, c satisfy

(i) a is decreasing;

(ii) b is locally $(2-\beta)/2$-Hölder continuous with $\beta \in (0, \beta_0)$;

(iii) c is increasing and locally $(\alpha-\beta)/\alpha$-Hölder continuous with $\beta \in (0, \beta_0)$.

Then, the PU holds.

Proof. From Corollary 1 and Lemma 7,

$$\mathbb{E}\left[|\Delta_{t\wedge T_N}|^\beta\right] \le K_4(\beta)\,\mathbb{E}\left[\int_0^{t\wedge T_N} |\Delta_s|^\beta\,ds\right],$$

and the required result follows from Gronwall's inequality. □

Case of $\alpha_- < \alpha_+$

Next, we shall prove the case of $\alpha_- < \alpha_+$.

Lemma 8

Let $\alpha_- < \alpha_+$. For $\beta \in (0, 1)$, it holds that

$$K_5(\beta) := \sup_{k \geq 0} \int_{\mathbb{R}_0} \left\{ |1 + k\, z|^\beta - 1 - \beta\, k\, z \right\} \nu_\rho^{\alpha_-, \alpha_+}(dz) < +\infty.$$

Proof. From Lemma 1, we have

$$\begin{aligned} &\int_{\mathbb{R}_0} \left\{ |1 + k\, z|^\beta - 1 - \beta\, k\, z \right\} \nu_\rho^{\alpha_-, \alpha_+}(dz) \\ &= k^{\alpha_+} \int_0^\infty \left\{ |1 + y|^\beta - 1 - \beta\, y \right\} \rho\left(\frac{y}{k} \right) |y|^{-\alpha_+ - 1}\, dy \\ &\quad + k^{\alpha_-} \int_{-\infty}^0 \left\{ |1 + y|^\beta - 1 - \beta\, y \right\} \rho\left(\frac{y}{k} \right) |y|^{-\alpha_- - 1}\, dy \\ &\to -\infty \quad \text{as } k \to +\infty. \quad \square \end{aligned}$$

Proof of our result ($\alpha_- < \alpha_+$)

Theorem 2 (Case of $\alpha_- < \alpha_+$)

Let $\alpha_- < \alpha_+$. Suppose that the coefficients a, b, c satisfy

(i) a is decreasing;

(ii) b is locally $(2-\beta)/2$-Hölder continuous with $\beta \in (0, 1)$;

(iii) c is increasing and locally $(\alpha_+ - \beta)/\alpha_+$-Hölder continuous with $\beta \in (0, 1)$.

Then, the PU holds.

Proof. From Corollary 1 and Lemma 8,

$$\mathbb{E}\left[|\Delta_{t\wedge T_N}|^\beta\right] \leq K_5(\beta)\,\mathbb{E}\left[\int_0^{t\wedge T_N} |\Delta_s|^\beta\, ds\right],$$

and the required result follows from Gronwall's inequality. □

References

- Fournier, N. (2013). On pathwise uniqueness for stochastic differential equations driven by stable Lévy processes. *Ann. Inst. H. Poincaré Probab. Statist.* 49(1): 138–159.
- Komatsu, T. (1982). On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations of jump type. *Proc. Japan Acad. Ser. A Math. Sci.* 58(8): 353–356.
- Li, Z., Mytnik, L. (2011). Strong solutions for stochastic differential equations with jumps. *Ann. Inst. H. Poincaré Probab. Statist.* 47(4): 1055–1067.
- Situ, R. (2005). *Theory of Stochastic Differential Equations with Jumps and Applications: Mathematical and Analytical Techniques with Applications to Engineering*. New York: Springer.
- Takeuchi, T., Tsukada, H. Remark on pathwise uniqueness of stochastic differential equations driven by Lévy processes. *Stoch. Anal. Appl.* to appear.
- Yamada, T., Watanabe, S. (1972). On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.* 11(1): 155–167.

Thank you for your attention.