On the pathwise uniqueness of solutions of SDEs driven by Lévy processes

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Let $W = \{W_t\,;\,t \ge 0\}$ be a 1-dimensional Brownian motion and $Z = \{Z_t\,;\,t \ge 0\}$ a 1-dimensional Lévy process characterized by the triplet $\left(\gamma_\rho^{\alpha_-,\alpha_+},\, 0,\, \nu_\rho^{\alpha_-,\alpha_+}(dz)\right)$ as follows:

$$\nu_\rho^{\alpha_-,\alpha_+}(dz) = \rho(z)\left(|z|^{-\alpha_- -1}\,\mathbb{I}_{(z<0)} + |z|^{-\alpha_+ -1}\,\mathbb{I}_{(z>0)}\right) dz, \quad \gamma_\rho^{\alpha_-,\alpha_+} = -\int_{|z|>1} z\,\nu_\rho^{\alpha_-,\alpha_+}(dz)$$

where $\rho : \mathbb{R}\setminus\{0\} \to [0,+\infty)$ is bounded and measurable such that

$$\rho(0+) = \lim_{z\to 0+} \rho(z) > 0, \quad \rho(0-) = \lim_{z\to 0-} \rho(z) \ge 0,$$

and $\alpha_-,\, \alpha_+ \in (1,2)$ such that $\alpha_- \le \alpha_+$. We shall consider the following stochastic differential equation (SDE):

$$X_t = x + \int_0^t a(X_s)\,ds + \int_0^t b(X_s)\,dW_s + \int_0^t c(X_{s-})\,dZ_s \tag{1}$$

where $a,\, b,\, c : \mathbb{R} \to \mathbb{R}$ are continuous with the linear growth condition. Then, the SDE (1) has a weak solution. In this talk, we shall study the pathwise uniqueness of solutions to the SDE (1).

Let us recall some known results on the pathwise uniqueness of solutions to the SDE (1) where Z is a strictly stable process of index $1 < \alpha < 2$ with parameters $(r_-,\, r_+)$, that is, $\rho(z) = r_-\,\mathbb{I}_{(z<0)} + r_+\,\mathbb{I}_{(z>0)}$ and $\alpha_- = \alpha_+ =: \alpha$.

- It is well-known that the pathwise uniqueness holds if $a,\, b,\, c$ are Lipschitz continuous.
- When $c = 0$, Yamada and Watanabe [5] have proved the pathwise uniqueness if a is locally Lipschitz continuous and b is locally $1/2$-Hölder continuous.
- When $a,\, b = 0$ and $r_- = r_+$, Komatsu [2] has done if c is locally $1/\alpha$-Hölder continuous.
- When $r_- = 0$, Li and Mytnik [3] have done if a is decreasing, b is locally $1/2$-Hölder continuous and c is increasing and locally $(\alpha-1)/\alpha$-Hölder continuous.
- When $b = 0$ and $r_- \le r_+$, Fournier [1] has done if a is decreasing and c is increasing and $(\alpha-\beta)/\alpha$-Hölder continuous where $\beta \in [\alpha-1,\, 1]$ satisfies the following equation: $\int_{\mathbb{R}\setminus\{0\}} \left\{|1+z|^\beta - 1 - \beta z\right\} \nu_\rho^{\alpha_-,\alpha_+}(dz) = 0$.

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Our class of driving processes includes parts of stable processes, truncated stable ones, tempered stable ones and relativistic stable ones. In this talk, we shall obtain the condition on the drift coefficient a, and the Hölder conditions on the diffusion coefficient b and the jump coefficient c, under which the pathwise uniqueness of the solutions can be justified.

Before introducing the main results, we shall prepare the notation: for $\alpha \in (1, 2)$ and $u \in [0, 1]$,

$$\beta(\alpha, u) := \frac{1}{\pi} \arccos \left[\frac{u^2 \sin^2(\pi\alpha) - (1 + u\cos(\pi\alpha))^2}{u^2 \sin^2(\pi\alpha) + \big(1 + u\cos(\pi\alpha)\big)^2} \right] \in [\alpha - 1, 1] .$$

Theorem 1 (Case of $\alpha_- = \alpha_+$) *Assume that $\rho(0-) \le \rho(0+)$. Let $\alpha_- = \alpha_+ =: \alpha$ and $\beta_0 := \beta(\alpha, \rho(0-)/\rho(0+))$. Suppose that the coefficients a, b, c of the SDE* (1) *are continuous with the linear growth order such that*

(i) the function a is decreasing;

(ii) the function b is locally $(2-\beta)/2$-Hölder continuous with $\beta \in (0, \beta_0)$;

(iii) the function c is increasing and locally $(\alpha-\beta)/\alpha$-Hölder continuous with $\beta \in (0, \beta_0)$.

Then, the pathwise uniqueness of the solutions to the SDE (1) *can be justified.*

Theorem 2 (Case of $\alpha_- < \alpha_+$) *Let $\alpha_- < \alpha_+$. Suppose that the coefficients a, b, c of the SDE* (1) *are continuous with the linear growth order such that*

(i) the function a is decreasing;

(ii) the function b is locally $(2-\beta)/2$-Hölder continuous with $\beta \in (0, 1)$;

(iii) the function c is increasing and locally $(\alpha_+ - \beta)/\alpha_+$-Hölder continuous with $\beta \in (0, 1)$.

Then, the pathwise uniqueness of the solutions to the SDE (1) *can be justified.*

References

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