# Uniqueness of solutions of infinite dimensional stochastic differential equations

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# Outline

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1 Osada, Tanemura : Infinite-dimensional stochastic differential equations and Tail $\sigma$-fields. arXiv:1412.8672v4

2 Kawamoto, Osada, Tanemura : Uniqueness of Dirichlet forms related to infinite systems of interacting Brownian motions. arXiv:1711.07796

## 1. Introduction

Infinite dimensional stochastic differential equations (ISDEs)

$\{B^i\}_{i\in\mathbb{N}}$ are independent $d$-dimensional Brownian motions.
$\Phi = \Phi(x)$ ; free potential.
$\Psi = \Psi(x, y)$ ; interaction potential.

We study ISDEs of $\mathbf{X} = (X^i)_{i\in\mathbb{N}} \in C([0,\infty); (\mathbb{R}^d)^{\mathbb{N}})$:

### Infinite dimensional stochastic differential equation

$$dX_t^i = dB_t^i - \frac{\beta}{2}\nabla_x\Phi(X_t^i)dt - \frac{\beta}{2}\sum_{j\neq i}^{\infty}\nabla_x\Psi(X_t^i, X_t^j)dt$$

$$(X_0^i)_{i\in\mathbb{N}} = \mathbf{s} = (s_i)_{i\in\mathbb{N}}$$

Existence ( solutions, strong solutions)
Uniqueness (in distribution, pathwise)

# 1. Introduction

Related results

1. Lang 1977,1978: $\Psi$ : smooth, with compact support
2. Fritz 1987: singular interaction
3. T. 1996, Fradon-Roelly-T. 2002: $\Psi$ : with hard core
4. Osada 2012: $\Psi$ : Ruelle's class, logarithmic (Dyson, Ginibre)
   Existence of solutions
5. Honda-Osada 2015 : logarithmic (Bessel)
   Existence and uniqueness of solutions
6. Osada-T. (arXiv1408.0632): logarithmic (Airy)
   Existence and uniqueness of solutions

## 2. Preliminaries (Notations)

$S$ : a subset of $\mathbb{R}^d$ s.t. $S_{\rm int}$ is connected, $\overline{S_{\rm int}} = S$, $|\partial S| = 0$.

$S^{\mathbb{N}}$ : the configuration space of labeled particles

The configuration space of unlabeled particles
$\mathfrak{M} = \{\xi = \sum \delta_{x^j} ; x^j \in S, \xi(K) < \infty$ for $\forall K$: compact$\}$

$\mathfrak{M}_{\rm s.i.} = \{\xi \in \mathfrak{M} : \xi(S) = \infty, \xi(\{x\}) \le 1$ for $\forall x \in S$ $\}$

label $\quad \mathfrak{l} : \mathfrak{M}_{\rm s.i.} \to S^{\mathbb{N}}, \qquad \mathfrak{l}(\sum_{j\in\mathbb{N}} \delta_{x^j}) = (x^j)_{j\in\mathbb{N}}$

unlabel $\quad \mathfrak{u} : (x^j)_{j\in\mathbb{N}} \to \sum_{j\in\mathbb{N}} \delta_{x^j}$

For $k \in \mathbb{N}$,
$\mathfrak{l}_{[k]} : \mathfrak{M}_{\rm s.i.} \to S^k \times \mathfrak{M}_{\rm s.i}, \qquad \mathfrak{l}_{[k]}(\sum_{j\in\mathbb{N}} \delta_{x^j}) = (x^1, \dots, x^k, \sum_{j>k} \delta_{x^j})$

$W(\mathcal{A}) = C([0,\infty) \to \mathcal{A})$, $W_a(\mathcal{A}) = \{w \in C([0,\infty) \to \mathcal{A}) : w(0) = a\}$

# 2. Preliminaries (ISDE1)

$\mathbf{X}_t = (X_t^j)_{j\in\mathbb{N}}$ : $S^{\mathbb{N}}$- valued continuous process

$\Xi_t = \sum_{j\in\mathbb{N}} \delta_{X_t^j}$, $\qquad \Xi_t^{\diamond k} = \sum_{j\in\mathbb{N}, j\neq k} \delta_{X_t^j}$

Let $\mathcal{H}$ and $\mathfrak{M}_{\mathrm{sde}}$ be Borel subsets of $\mathfrak{M}$ s.t. $\quad \mathcal{H} \subset \mathfrak{M}_{\mathrm{sde}} \subset \mathfrak{M}_{s.i.}$

and put $\qquad \mathbf{H} = \mathfrak{l}(\mathcal{H}), \quad \mathbf{S}_{\mathrm{sde}} = \mathfrak{u}^{-1}(\mathfrak{M}_{\mathrm{sde}}), \quad \mathbf{S}_{\mathrm{sde}}^{[1]} = \mathfrak{u}_{[1]}^{-1}(\mathfrak{M}_{\mathrm{sde}})$

Let $\sigma : \mathbf{S}_{\mathrm{sde}}^{[1]} \to \mathbb{R}^{d^2}$, $\ b : \mathbf{S}_{\mathrm{sde}}^{[1]} \to \mathbb{R}^d$ be Borel functions, $\quad \mathbf{x} \in \mathbf{H}$.

$$\begin{aligned} dX_t^j &= \sigma(X_t^j, \Xi_t^{\diamond j}) dB_t^j + b(X_t^j, \Xi_t^{\diamond j}) dt \\ \mathbf{X} &\in W(\mathbf{S}_{\mathrm{sde}}) \\ \mathbf{X}_0 &= \mathbf{x} \end{aligned} \tag{ISDE 1}$$

Here $\mathbf{B} = (B^j)_{j\in\mathbb{N}}$ is a $(\mathbb{R}^d)^{\mathbb{N}}$-valued Brownian motion.

# 2. Preliminaries (a solutions)

## Definition (a solution of (ISDE1))

We call $(\mathbf{X}, \mathbf{B})$ a solution of (ISDE1) if $(\mathbf{X}, \mathbf{B})$ is $S^{\mathbb{N}} \times (\mathbb{R}^d)^{\mathbb{N}}$-valued process defined on $(\Omega, \mathfrak{F}, P)$ with a reference family $\{\mathfrak{F}_t\}_{t\geq 0}$ s.t.

(i) $\mathbf{X}$ is an $\mathfrak{F}_t$-adapted continuous process.

(ii) $\mathbf{B}$ is $(\mathbb{R}^d)^{\mathbb{N}}$-valued $\mathfrak{F}_t$-Brownian motion with $\mathbf{B}_0 = \mathbf{0}$.

(iii) $\sigma(X_t^j, \Xi_t^{\diamond j})$ and $b(X_t^j, \Xi_t^{\diamond j})$ are $\mathfrak{F}_t$-adapted processes belonging to $\mathcal{L}^2$ and $\mathcal{L}^2$, respectively.

(iv) $(\mathbf{X}, \mathbf{B})$ satisfies for all $t$ and for a.s. $\omega$

$$X_t^j = x^j + \int_0^t \sigma(X_u^j, \Xi_u^{\diamond j}) dB_u^j du + \int_0^t b(X_u^j, \Xi_u^{\diamond j}) du, \quad j \in \mathbb{N}$$

# 2. Preliminaries (a strong solution)

## Definition

1. We call a solution $(\mathbf{X}, \mathbf{B})$ of (ISDE1) a strong solution if there exists a function $\mathbb{F}_{\mathbf{x}} : W_0((\mathbb{R}^d)^{\mathbb{N}}) \to W(S^{\mathbb{N}})$ such that $\mathbf{X} = \mathbb{F}_{\mathbf{x}}(\mathbf{B})$.
$\mathbb{F}_{\mathbf{x}}$ is also called a strong solution.
(For any Brownian motion $\mathbf{B}$, $\mathbf{X} = \mathbb{F}_{\mathbf{x}}(\mathbf{B})$ is a solution of (ISDE1))

2. We call (ISDE1) has a unique strong solution if a strong solution $\mathbb{F}_{\mathbf{x}}$ of (ISDE1) exists and any solution $(\mathbf{X}', \mathbf{B}')$ of (ISDE1) satisfies $\mathbf{X}' = \mathbb{F}_{\mathbf{x}}(\mathbf{B}')$.

3. We call the pathwise uniqueness of solutions for (ISDE1) holds if $\mathbf{X} = \mathbf{X}'$ a.s. for any solutions $\mathbf{X}$ and $\mathbf{X}'$ on the same probability space with the same Brownian motion $\mathbf{B}$.

# 3. First tail theorem (Assumption **(B1)**)

## Assumption **(B1)**

(ISDE1) has a solution $(\mathbf{X}, \mathbf{B})$.

Let $(\mathbf{X}, \mathbf{B})$ be a solution of (ISDE1). For $m \in \mathbb{N}$ we put

$$\mathbf{X}^m = (X^1, X^2, \dots, X^m), \quad \mathbf{X}^{m*} = (X^{m+1}, X^{m+2}, \dots).$$

and

$$\Xi^{m*} = \mathfrak{u}(\mathbf{X}^{m*}) = \sum_{j=m+1}^{\infty} \delta_{X^j}$$

We define $\sigma_{\mathbf{X}}^m : [0, \infty) \times S^m \to (\mathbb{R}^{d^2})^{\mathbb{N}}$, $b_{\mathbf{X}}^m : [0, \infty) \times S^m \to (\mathbb{R}^d)^{\mathbb{N}}$ by

$$\sigma_{\mathbf{X}}^m(t, (u, \mathbf{v})) = \sigma(u, \mathfrak{u}(\mathbf{v}) + \Xi_t^{m*}),\ b_{\mathbf{X}}^m(t, (u, \mathbf{v})) = b(u, \mathfrak{u}(\mathbf{v}) + \Xi_t^{m*})$$

# 3. First tail theorem (SDE-$m$)

We consider the infinite system of finite dimensional SDEs (SDE-m) associated with (ISDE1):

$$\begin{aligned} &dY_t^{m,j} = \sigma_{\mathbf{X}}^m(t,(Y_t^{m,j},\mathbf{Y}_t^{m,j\diamond}))dB_t^j + b_{\mathbf{X}}^m(t,(Y_t^{m,j},\mathbf{Y}_t^{m,j\diamond}))dt, \\ &\mathbf{Y}_t^m \in \mathbf{S}_{\mathrm{sde}}(t,\mathbf{X}), \quad \text{for } \forall t \ge 0, \\ &\mathbf{Y}_0^m = \mathbf{x}^m = (x^1,x^2,\dots,x^m), \end{aligned} \tag{SDE-$m$}$$

where

$$\mathbf{S}_{\mathrm{sde}}(t,\mathbf{X}) = \{\mathbf{x}^m \in S^m : \mathfrak{u}(\mathbf{x}^m) + \Xi_t^{m*} \in \mathfrak{M}_{\mathrm{sde}}\}$$

and

$$\mathbf{Y}^{m,j\diamond} = (Y^{m,1},Y^{m,2},\dots,Y^{m,j-1},Y^{m,j+1},\dots,Y^{m,m})$$

for $\mathbf{Y}^m = (Y^{m,1},Y^{m,2},\dots,Y^{m,m})$.

# 3. First tail theorem (Assumption **(B2)**, IFC)

## Assumption **(B2)**

For each $m \in \mathbb{N}$, (SDE-$m$) has a strong solution $\mathbf{Y}^m$, and the pathwise uniqueness for solutions holds for each $m \in \mathbb{N}$.

Assume **(B1)** and **(B2)**. Put

$$F_{\mathbf{x}}^m(\mathbf{B}, \mathbf{X}) = (\mathbf{Y}^m, \mathbf{X}^{m*}) = (Y^{m,1}, \dots, Y^{m,m}, X^{m+1}, X^{m+2}, \dots).$$

From the pathwise uniqueness of solution to (SDE-$m$) for each $m$

## Facts

1. $F_{\mathbf{x}}^{m,i}(\mathbf{B}, \mathbf{X}) = F_{\mathbf{x}}^{m+1,i}(\mathbf{B}, \mathbf{X}) \quad \forall m \in \mathbb{N}, \forall i = 1, 2, \dots, m.$
2. $(F_{\mathbf{x}}^{\infty}(\mathbf{B}, \mathbf{X}), \mathbf{B}) = (\mathbf{X}, \mathbf{B}) \quad P\text{-a.s.}$

IFC (infinite system of finite-dimensional SDEs with consistency)

# 3. First tail theorem (IFC solution)

## Definition (IFC solution)

A probability measure $\overline{P}_{\mathbf{x}}$ on $W(S^{\mathbb{N}}) \times W_{\mathbf{0}}((\mathbb{R}^d)^{\mathbb{N}})$ is called an (asymptotic) IFC solution for (ISDE1) if $\overline{P}_{\mathbf{s}}$ satisfies

1. $\overline{P}_{\mathbf{x}}(\mathbf{B} \in \cdot) = P^{\infty}_{Br}(\cdot) :=$ the distribution of $(\mathbb{R}^d)^{\mathbb{N}}$-valued BM
2. For $\forall j \in \mathbb{N}$ for $\overline{P}_{\mathbf{x}}$-a.s. $(\mathbf{X}, \mathbf{B})$

$$\lim_{m\to\infty} F_{\mathbf{x}}^{m,j}(\mathbf{X}, \mathbf{B}) = F_{\mathbf{x}}^{\infty,j}(\mathbf{X}, \mathbf{B})$$

$$\lim_{m\to\infty} \int_0^{\cdot} \sigma^j(F_{\mathbf{x}}^{m,j}(\mathbf{X}, \mathbf{B}))_u dB_u^j = \int_0^{\cdot} \sigma^j(F_{\mathbf{x}}^{\infty,j}(\mathbf{X}, \mathbf{B}))_u dB_u^j$$

$$\lim_{m\to\infty} \int_0^{\cdot} b^i(F_{\mathbf{x}}^{m}(\mathbf{X}, \mathbf{B}))_u du = \int_0^{\cdot} b^i(F_{\mathbf{x}}^{\infty}(\mathbf{X}, \mathbf{B}))_u du \quad \text{in } W(\mathbb{R}^d),$$

where

$$\sigma^j(\mathbf{X})_t = \sigma(X_t^j, \Xi_t^{\diamond j}), \qquad b^j(\mathbf{X})_t = b(X_t^j, \Xi_t^{\diamond j}).$$

# 3. First tail theorem (IFC solution)

**Remark**

In the definition of IFC solution $(\mathbf{X}, \mathbf{B})$ under $\overline{P}_{\mathbf{s}}$ is not always a solution of (ISDE1). $(\mathbf{X}, \mathbf{B})$ is a solution, in case $F_{\mathbf{x}}^{\infty}(\mathbf{X}, \mathbf{B}) = \mathbf{X}$,

## Facts

3. Assume **(B1)** and **(B2)**.
Let $\overline{P}_{\mathbf{x}}$ be the distribution of a solution of (ISDE1).
Then $\overline{P}_{\mathbf{x}}$ is an IFC solution for (ISDE1).

4. Assume **(B2)**.
Let $\overline{P}_{\mathbf{x}}$ be an IFC solution for (ISDE1).
Then $(F_{\mathbf{x}}^{\infty}(\mathbf{X}, \mathbf{B}), \mathbf{B})$ is a solution of (ISDE1) under $\overline{P}_{\mathbf{x}}$.

# 3. First tail theorem (Tail $\sigma$-fields)

$\mathcal{T}_{path}(S^{\mathbb{N}})$ is the tail $\sigma$-field of $\mathbf{W}$ defined by

$$\mathcal{T}_{path}(S^{\mathbb{N}}) = \bigcap_{m=1}^{\infty} \sigma(\mathbf{X}^{m*}), \text{ where } \mathbf{X}^{m*} = (X^i)_{i=m+1}^{\infty}.$$

For a probability measure $P$ such that $\mathcal{T}_{path}(S^{\mathbb{N}})$ is $P$-trivial, that is,

$$P(A) \in \{0,1\}, \forall A \in \mathcal{T}_{path}(S^{\mathbb{N}}),$$

we set

$$\mathcal{T}_{path}^{[1]}(S^{\mathbb{N}}; P) = \{A \in \mathcal{T}_{path}(S^{\mathbb{N}}); P(A) = 1\}.$$

## 3. First tail theorem (Tail $\sigma$-fields and measurability)

For the distribution $\overline{P}_{\mathbf{x}}$ of a solution of (ISDE1), put

$$\overline{P}_{\mathbf{x}.\mathbf{B}}(\cdot) = \overline{P}_{\mathbf{x}}(\cdot|\mathbf{B})$$

### Fact

5. The map $F_{\mathbf{x}}^m$ is $\overline{\sigma(\mathbf{X}^{m*}) \times \mathcal{B}(W_0((\mathbb{R}^d)^{\mathbb{N}})}^{\overline{P}_{\mathbf{x}}}$ -measurable.
6. The map $F_{\mathbf{x}}^\infty$ is $\displaystyle\bigcap_{m\in\mathbb{N}} \overline{\sigma(\mathbf{X}^{m*}) \times \mathcal{B}(W_0((\mathbb{R}^d)^{\mathbb{N}})}^{\overline{P}_{\mathbf{x}}}$ -measurable.
7. The map $F_{\mathbf{x}}^\infty$ is $\overline{\mathcal{T}_{path}(S^{\mathbb{N}})}^{\overline{P}_{\mathbf{x},\mathbf{B}}}$ -measurable for $P_{Br}^\infty(\cdot)$-a.s. $\mathbf{B}$.

## 3. First tail theorem (Theorem 1)

$\overline{P}_{\mathbf{x},\mathbf{B}}(\cdot) = \overline{P}_{\mathbf{x}}(\cdot|\mathbf{B})$ : the regular conditional probability.

### Assumptions **(B3)** – **(B5)**

**(B3)** $\mathcal{T}_{path}(S^{\mathbb{N}})$ is $\overline{P}_{\mathbf{x},\mathbf{B}}$-trivial for $P^{\infty}_{Br}$- a.s. $\mathbf{B}$.

**(B4)** $\mathcal{T}^{[1]}_{path}(S^{\mathbb{N}}; \overline{P}_{\mathbf{x},\mathbf{B}}) = \mathcal{T}^{[1]}_{path}(S^{\mathbb{N}}; \overline{P'}_{\mathbf{x},\mathbf{B}})$ for $P^{\infty}_{Br}$- a.s. $\mathbf{B}$.

**(B5)** $\mathcal{T}^{[1]}_{path}(S^{\mathbb{N}}; \overline{P}_{\mathbf{x},\mathbf{B}})$ is independent of $\overline{P}_{\mathbf{x},\mathbf{B}}$ for $P^{\infty}_{Br}$- a.s. $\mathbf{B}$.

### Theorem 1 (First tail theorem)

1. **(B1)–(B3)** $\Rightarrow$ (ISDE1) has a strong solution.
2. **(B1)–(B4)** $\Rightarrow$ Strong solutions $\mathbf{X}$ and $\mathbf{X}'$ satisfy $\mathbf{X} = \mathbf{X}'$ a.s.
3. **(B1)–(B5)** $\Rightarrow$ (ISDE1) has a unique strong solution.

# 4. Second tail theorem (Tail $\sigma$-field on $\mathfrak{M}$)

Unlabeled configuration space on $S$

$$\mathfrak{M} = \{\xi = \sum_j \delta_{x_j} : \xi(K) < \infty \quad \forall K : compact\}$$

$\mathfrak{M}$ is Polish with the vague topology. Set

$$\mathfrak{M}_{\mathrm{s.i.}} = \{\xi \in \mathfrak{M} : \xi(S) = \infty,\ \mathfrak{s}(\{x\}) \in \{0,1\},\ \forall x \in S\}.$$

Tale $\sigma$-field $\mathcal{T}(\mathfrak{M})$ on $\mathfrak{M}$ is given by

$$\mathcal{T}(\mathfrak{M}) = \bigcap_{r=1}^{\infty} \sigma(\pi_r^c)$$

where $\pi_r^c(\xi)(\cdot) = \xi(\cdot \cap S_r^c)$, $\quad S_r = \{x \in S : |x| \le r\}$.

# 4. Second tail theorem (Labeled map on the path space)

Labeled maps on $\mathfrak{M}$ and $W(\mathfrak{M}_{\mathrm{s.i.}})$

A map $\mathfrak{l} : \mathfrak{M} \to S^{\mathbb{N}}$ given as

$$\mathfrak{l}(\xi) = \mathbf{x} = (x^j)_{j\in\mathbb{N}}, \text{ for } \xi = \sum_{j=1}^{\infty} \delta_{x^j} \in \mathfrak{M}$$

is called a label. For a label $\mathfrak{l}$, we can deternine the map $\mathfrak{l}_{path}$

$$\mathfrak{l}_{path} : W(\mathfrak{M}_{\mathrm{s.i.}}) \to W(S^{\mathbb{N}})$$

such that

$$\mathfrak{l}(\Xi)_0 = \mathbf{X_0} = (X_0^j)_{j\in\mathbb{N}}, \text{ for } \Xi = \sum_{j=1}^{\infty} \delta_{X^m} \in W(\mathfrak{M}_{\mathrm{s.i.}})$$

For a prob. meas.

$$\mathbb{P}_\mu = \int_{\mathfrak{M}} \mu(d\xi)\mathbb{P}_\xi \quad \text{ on } W(\mathfrak{M}_{\mathrm{s.i.}}) \quad (\text{ i.e. } \mathbb{P}_\mu \circ \Xi_0^{-1} = \mu),$$

put

$$\mu^{\mathfrak{l}} = \mu \circ \mathfrak{l}^{-1}, \quad P_{\mu^{\mathfrak{l}}} = \mathbb{P}_\mu \circ \mathfrak{l}_{path}^{-1}$$

## 4. Second tail theorem (Theorem 2)

### Assumptions **(C1)** – **(C3)**

**(C1)** $\mathcal{T}(\mathfrak{M})$ is $\mu$-trivial.

**(C2)** $\mathbb{P}_{\mu} \circ {\Xi_t}^{-1} \prec \mu$, $\forall t \in [0, T]$ ($\mu$-**AC**) : $\mu$-absolutely cont. cond.

**(C3)** $P_{\mu^{\mathfrak{l}}}(\bigcap_{r=1}^{\infty}\{m_r(\mathbf{X}) < \infty\}) = 1$ (**NBJ**) : No big jump condition

where $m_r(\mathbf{X}) = \inf\{m \in \mathbb{N} : X_t^n \in S_r^c,\ \forall t \in [0, T],\ \forall n > m\}$.

### Theorem 2 (Second tail theorem)

Assume **(B2)**. Suppose that there exists $\mathbb{P}_{\mu}$ satisfying **(C1)** – **(C3)**, and

$$P_{\mu^{\mathfrak{l}}}(F_{\mathbf{x}}^{\infty}(\mathbf{X}, \mathbf{B}) = \mathbf{X}) = 1.$$

Then **(B1)** and **(B3)** – **(B5)** hold for $\mu^{\mathfrak{l}}$-a.s. $\mathbf{x}$.

# 4. Second tail theorem (Outline of the proof Theorem 2)

$$\begin{array}{ccccccc} \mathcal{T}(\mathfrak{M}) & \xrightarrow{\text{Step I}} & \tilde{\mathcal{T}}_{path}(\mathfrak{M}) & \xrightarrow{\text{Step II}} & \tilde{\mathcal{T}}_{path}(S^{\mathbb{N}}) & \xrightarrow{\text{Step III}} & \mathcal{T}_{path}(S^{\mathbb{N}}) \\ \mu & & \mathbb{P}_{\mu} & & P_{\mu^{\mathfrak{l}}} & & \overline{P}_{\mathbf{x},\mathbf{B}} \end{array}$$

Here, $\tilde{\mathcal{T}}_{path}(\mathfrak{M})$ is the cylindrical tail $\sigma$-field on $W(\mathfrak{M})$ defined as

$$\tilde{\mathcal{T}}_{path}(\mathfrak{M})) = \bigvee_{\mathbf{t}=(t_1,t_2,\dots,t_n),n\in\mathbb{N}} \bigcap_{r=1}^{\infty} \sigma[\pi_r^c(\Xi_{t_i}), 1 \le i \le n].$$

and $\tilde{\mathcal{T}}_{path}(S^{\mathbb{N}})$ is the cylindrical tail $\sigma$-field on $W(S^{\mathbb{N}})$ defined as

$$\tilde{\mathcal{T}}_{path}(S^{\mathbb{N}}) = \bigvee_{\mathbf{t}=(t_1,t_2,\dots,t_n),n\in\mathbb{N}} \bigcap_{m=1}^{\infty} \sigma[\mathbf{X}_{t_i}^{m*}, 1 \le i \le n].$$

# 5. Examples

1. Lennard-Jones 6-12 potential
   $d = 3$ and $\Psi_{6,12}(x) = \{|x|^{-12} - |x|^{-6}\}$.

$$dX_t^j = dB_t^j + \frac{\beta}{2} \sum_{k \in \mathbb{N}, k \neq j} \{ \frac{12(X_t^j - X_t^j)}{|X_t^j - X_t^k|^{14}} - \frac{6(X_t^j - X_t^k)}{|X_t^j - X_t^k|^8} \} dt \quad (j \in \mathbb{N})$$

2. Riesz potentials
   $d < a \in \mathbb{N}$ and $\Psi_a(x) = (\beta/a)|x|^{-a}$.

$$dX_t^j = dB_t^j + \frac{\beta}{2} \sum_{k \in \mathbb{N}, k \neq j} \frac{X_t^j - X_t^k}{|X_t^j - X_t^k|^{a+2}} dt \quad (j \in \mathbb{N})$$

## 5. Examples

logarithmic potential $\Psi(x, y) = \log|x - y|$

3. Sine RPF: $\beta = 1, 2, 4,\ d = 1.$

$$dX_t^j = dB_t^j + \lim_{L\to\infty}\left\{\frac{\beta}{2}\sum_{k\neq j, |X_t^k|<L}\frac{1}{X_t^j - X_t^k}\right\}dt \quad (j\in\mathbb{N})$$

4. Airy RPF : $\beta = 1, 2, 4,\ d = 1.$

$$dX_t^j = dB_t^j + \lim_{L\to\infty}\left\{\frac{\beta}{2}\sum_{k\neq j, |X_t^k|<L}\frac{1}{X_t^j - X_t^k} - \int_{|x|<L}\frac{\widehat{\rho}(x)dx}{-x}\right\}dt$$

$$(j\in\mathbb{N})$$

where $\widehat{\rho}(x) = \frac{1}{\pi}\sqrt{-x}\mathbf{1}(x<0)$ である.

# 5. Examples

3.3 Ginibre RPF : $d = 2$.

$$dX_t^j = dB_t^j + \lim_{L\to\infty} \sum_{k\neq j,\ |X_t^j - X_t^k|<L} \frac{X_t^j - X_t^k}{|X_t^j - X_t^k|^2} dt \quad (j \in \mathbb{N})$$

$$dX_t^j = dB_t^j - X_t^j dt + \lim_{L\to\infty} \sum_{k\neq j,\ |X_t^k|<L} \frac{X_t^j - X_t^k}{|X_t^j - X_t^k|^2} dt \quad (j \in \mathbb{N})$$

# 6. Condition (B1)

(B.1) (ISDE1) has a solution $(\mathbf{X}, \mathbf{B})$

$$dX_t^j = dB_t^j - \frac{\beta}{2}\nabla_x \Phi(X_t^j)dt - \frac{\beta}{2}\sum_{k\neq j}^{\infty} \nabla_x \Psi(X_t^i, X_t^j)dt \ (j \in \mathbb{N})$$

$$(X_0^i)_{i\in\mathbb{N}} = (x^i)_{i\in\mathbb{N}} = \mathbf{x}$$

(1) Construction of $\mathfrak{M}$-valued process $\Xi_t = \sum_{j\in\mathbb{N}} \delta_{X_t^i}$ by Dirichlet form $(\mathcal{E}^\mu, \mathcal{D}^\mu)$ associated with a $(\Phi, \Psi)$-quasi-Gibbs state $\mu$. [Osada AOP 2013].

(2) Existence of solutions of (ISDE1) related to the logarithmic derivative of the Campbell measure $\mu^{[1]}$ of $\mu$ [Osada PTRF 2012].

# 6. Condition (B1) (Quasi Gibbs state)

Hamiltonian for $\Phi$, $\Psi$ on $S_\ell = \{x \in \mathbb{R}^d : |x| \le \ell\}$

$$H_\ell(\zeta) = \sum_{x \in \mathrm{supp}\zeta \cap S_\ell} \Phi(x) + \sum_{x,y \in \mathrm{supp}\zeta \cap S_\ell, x \neq y} \Psi(x,y),$$

## Definition(Quasi Gibbs state)

A RPF $\mu$ is called a $(\Phi, \Psi)$-quasi Gibbs state, if

$$\mu^m_{\ell,\xi}(d\zeta) = \mu(d\zeta | \pi_{S_\ell^c}(\xi) = \pi_{S_r^c}(\zeta), \zeta(S_\ell) = m),$$

satisfies that for $\ell, m, k \in \mathbb{N}$, $\mu$-a.s. $\xi$

$$c^{-1} e^{-H_\ell(\zeta)} \Lambda^m_\ell(d\zeta) \le \mu^m_{\ell,\xi}(\pi_{S_\ell} \in d\zeta) \le c e^{-H_\ell(\zeta)} \Lambda^m_\ell(d\zeta)$$

where $c = c(\ell, m, \xi) > 0$ is a constant depending on $\ell, m, \xi$,
$\Lambda^m_\ell$ is the rest. of PRF with int. meas. $dx$ on $\mathfrak{M}^m_\ell = \{\xi(S_\ell) = m\}$.

# 6. Condition (B1) (Polynomial functions)

A function $f$ on $\mathfrak{M}$ is called a polynomial function if it is represented as

$$f(\xi) = Q\left(\langle \varphi_1, \xi\rangle, \langle \varphi_2, \xi\rangle, \dots, \langle \varphi_\ell, \xi\rangle\right)$$

with a polynomial function $Q$ on $\mathbb{R}^\ell$, and smooth functions $\phi_j$, $1 \le j \le \ell$, with compact supports, where

$$\langle \varphi, \xi\rangle = \int_{\mathbb{R}^d} \varphi(x)\xi(dx).$$

We denote by $\mathcal{P}$ the set of all polynomial functions on $\mathfrak{M}$. A polynomial function is local and smooth: $\exists K$ compact s.t.

$$f(\xi) = f(\pi_K(\xi)) \quad \text{and} \quad f(\xi) = f(x_1, \dots, x_n) \text{ is smooth}$$

where $n = \xi(K)$ and $\pi_K(\xi)$ is the restriction of $\xi$ on $K$.

# 6. Condition (B1) (Square fields)

For $f \in \mathcal{P}$ we introduce the square field on $\mathfrak{M}$ defined by

$$\mathbf{D}(f,g)(\xi) = \frac{1}{2}\int_{\mathbb{R}^d} \xi(dx)\nabla_x f(\xi)\cdot\nabla_x g(\xi).$$

For a RPF (a probability measure $\mu$ on $\mathfrak{M}$), we introduce the bilinear form on $L^2(\mu)$ defined by

$$\mathcal{E}^\mu(f,g) = \int_{\mathfrak{M}} \mathbf{D}(f,g)(\xi)\mu(d\xi), \quad f,g \in \mathcal{D}_\circ^\mu$$
$$\mathcal{D}_\circ^\mu = \{f \in \mathcal{P}\cap L^2(\mathcal{M},\mu) : \| f \|_1 < \infty\}.$$

where

$$\| f \|_1^2 = \| f \|_{L^2(\mu)}^2 + \mathcal{E}^\mu(f,f).$$

# 6. Condition (B1) (Systems of unlabeled particles)

We make assumptions on RPF $\mu$

**(A1)** $\mu$ is a $(\Phi, \Psi)$-quasi Gibbs state, and $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ and $\Psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ satisfy

$$c^{-1}\Phi_0(x) \le \Phi(x) \le c\ \Phi_0(x)$$
$$c^{-1}\Psi_0(x-y) \le \Psi(x,y) \le c\ \Psi_0(x-y)$$

for some $c > 1$ and locally bounded from below and lower semi-continuous function $\Phi_0, \Psi_0$ with $\{x \in \mathbb{R}^d : \Psi_0(x) = \infty\}$ being compact.

Let $k \in \mathbb{N}$.

**(A2)** $\displaystyle\sum_{n=1}^{\infty} n^k \mu(\mathfrak{M}_r^n) = \int_{\mathfrak{M}} \xi(S_r)^k \mu(d\xi) < \infty$, $\forall r \in \mathbb{N}$, $\forall k \in \mathbb{N}$.

# 6. Condition (B1) (Systems of unlabeled particles 2)

## Proposition 1 (Osada 13)

Suppose that **(A1)** – **(A2)**. Then

1. $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}_{\circ})$ is closable on $L^2(\mathfrak{M}, \mu)$, and the closure $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu})$ is a quasi-regular Dirichlet form.
2. The associated diffusion process $(\Xi_t, \mathbb{P}_{\xi})$ can be constructed and it has $\mu$ as a reversible probability measure.

**Remark.** In Osada[CMP 1996, AOP 2013], the set $\mathcal{D}_{\infty}^{\mathrm{local}}$ of local smooth functions are introduced and set

$$\mathcal{D}^{\mu}_{\circ} = \{ f \in \mathcal{D}_{\infty}^{\mathrm{local}} \cap L^2(\mathcal{M}, \mu) : \| f \|_1 < \infty \}.$$

Although $\mathcal{P} \subset \mathcal{D}^{\mu}_{\circ}$, it give the same Dirichlet form $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu})$. Osada-T. [PJA 2014]

# 6. Condition (B1) (Logarithmic derivative)

## Definition (log derivative)

We call $\mathbf{d}^{\mu} \in L^1_{loc}(\mathbb{R}^d \times \mathfrak{M}, \mu^{[1]})$ the logarithmic derivative of $\mu$ if

$$\int_{\mathbb{R}^d \times \mathfrak{M}} \mathbf{d}^{\mu}(x,\eta)\varphi(x,\eta)d\mu^{[1]}(x,\eta) = -\int_{\mathbb{R}^d \times \mathfrak{M}} \nabla_x \varphi(x,\eta)d\mu^{[1]}(x,\eta),$$

is satisfied for $\varphi \in C_c^{\infty}(\mathbb{R}^d) \otimes \mathcal{D}_{\infty}^{\mathrm{loc}}$.

Here $\mu^{[k]}$, $k \in \mathbb{N}$ is the Campbell measure of $\mu$:

$$\mu^{[k]}(A \times B) = \int_A \mu_{\mathbf{x}}(B)\rho^k(\mathbf{x})d\mathbf{x}, \quad A \in \mathcal{B}((\mathbb{R}^d)^k), B \in \mathcal{B}(\mathfrak{M}).$$

and $\mu_{\mathbf{x}}$ is the reduced Palm measure conditioned at $\mathbf{x} \in (\mathbb{R}^d)^k$

$$\mu_{\mathbf{x}} = \mu(\cdot - \sum_{j=1}^{k} \delta_{x^j} \Big| \xi(x^j) \geq 1 \text{ for } j = 1, 2, \dots, k),$$

# 6. Condition (B1) (ISDE representation)

Let $\mathbf{X} = \mathfrak{l}_{\mathrm{path}}(\Xi_t) = (X_t^j)_{j\in\mathbb{N}}$. We make the following assumptions:

**(A3)** $\{X^j\}$ do not collide each other.

**(A4)** each particle $X^j$ never explodes.

**(A5)** $\mu$ has a logarithmic derivative $\mathbf{d}^\mu$.

## Proposition 2 [Osada 2010, 2012]

Assume **(A1)**-**(A5)**. Then there exists $\mathcal{H} \subset \mathfrak{M}$ such that $\mu(\mathcal{H}) = 1$, and for any $\xi \in \mathcal{H}$, there exists $(\mathbb{R}^d)^{\mathbb{N}}$-valued continuous process $\mathbf{X}_t = (X_t^j)_{j=1}^\infty$ satisfying $\mathbf{X}_0 = \mathfrak{l}(\xi) = \mathbf{x} = (x^j)_{j=1}^\infty$ and

$$dX_t^j = dB_t^j + \frac{1}{2}\mathbf{d}^\mu\left(X_t^j, \sum_{k:k\neq j} \delta_{X_t^k}\right)dt, \quad j \in \mathbb{N}. \qquad \text{(ISDE1)}$$

# 6. Condition (B1) (Tagged particle processes)

For $\xi \in \mathcal{H}$ let $\mathfrak{l}(\xi) = \mathbf{x} = (x^j)_{j=1}^{\infty}$, $\xi^{m*} = \sum_{j \geq m+1} \delta_{x^j}$

0. $(\Xi_t, \mathbb{P}_{\xi}) \Longleftrightarrow (\mathcal{E}^{\mu}, \mathcal{D}^{\mu})$
1. $((X_t^1, \Xi_t^{1*}), \mathbb{P}_{(x^1, \xi^{1*})}) \Longleftrightarrow (\mathcal{E}^{\mu^{[1]}}, \mathcal{D}^{\mu^{[1]}})$
2. $((\mathbf{X}_t^2, \Xi_t^{2*}), \mathbb{P}_{((x^1, x^2), \xi^{2*})}) \Longleftrightarrow (\mathcal{E}^{\mu^{[2]}}, \mathcal{D}^{\mu^{[2]}})$

m. $((\mathbf{X}_t^m, \Xi_t^{m*}), \mathbb{P}_{((x^1, x^2, \dots, x^m), \xi^{m*})}) \Longleftrightarrow (\mathcal{E}^{\mu^{[m]}}, \mathcal{D}^{\mu^{[m]}})$

We construct the sequence

$$\{(\mathbf{X}_t^m, \Xi_t^{m*})\}_{k \in \mathbb{N}}$$

with the consistency : $\forall m \in \mathbb{N}$

$$X_t^{m,j} = X_t^{m+1,j} \equiv X_t^j, \quad 1 \leq j \leq m, \quad \Xi_t^{m*} = \delta_{X_t^{m+1,m+1}} + \Xi_t^{m+1*}$$

Then $\mathbf{X}_t = (X_t^1, X_t^2, \dots)$ is a Dirichlet process.

# 7. Condition (B3) – (B5)

We check Assumptions in Theorem 2 (Second tail theorem):

## Assumptions (C1) – (C3)

**(C1)** $\mathcal{T}(\mathfrak{M})$ is $\mu$-trivial.

**(C2)** $\mathbb{P}_{\mu} \circ \Xi_t{}^{-1} \prec \mu$, $\forall t \in [0, T]$ ($\mu$-**AC**) : $\mu$-absolutely cont. cond.

**(C3)** $P_{\mu^{\mathfrak{l}}}(\bigcap_{r=1}^{\infty}\{m_r(\mathbf{X}) < \infty\}) = 1$ (**NBJ**) : No big jump condition

where $m_r(\mathbf{X}) = \inf\{m \in \mathbb{N} : X_t^n \in S_r^c,\ \forall t \in [0, T],\ \forall n > m\}$.

Let $(\mathbf{X}, \mathbf{B})$ be a diffusion process constructed by Propositions 1 and 2.

1. $(\Xi_t, \mathbb{P}_{\mu})$ is reversible $\Longrightarrow$ **(C2)**
2. Lyons-Zheng decomposition $\Longrightarrow$ **(C3)**

If **(C1)** holds, $(\mathbf{X}, \mathbf{B})$ under $P_{\mathbf{x}}$ satisfies **(B3)-(B5)** for $\mu^{\mathfrak{l}}$-a.s. $\mathbf{x}$.

# Lyons-Zheng decomposition

$(\Xi_t^{[m]}, \mathbb{P}_{(x,\xi)}^{[m]})$ : the process with reversible measure $\mu^{[m]}$

$G(\Xi_t^{[m]})$ : Dirichlet process

$G(\Xi_t^{[m]}) - G(\Xi_0^{[m]}) = M(\Xi^{[m]})_t + \int_0^t b(\Xi_s^{[m]})ds$ Fukushima decomposition

For fixed $T > 0$, put $\tilde{\Xi}_t^{[m]} := \Xi_{T-t}^{[m]}$.

$$
\begin{aligned}
G(\Xi_t^{[m]}) - G(\Xi_0^{[m]}) &= G(\tilde{\Xi}_{T-t}^{[m]}) - G(\tilde{\Xi}_T^{[m]}) \\
&= M(\tilde{\Xi}^{[m]})_{T-t} - M(\tilde{\Xi}^{[m]})_T + \int_T^{T-t} b(\Xi_{T-s}^{[m]})ds
\end{aligned}
$$

$$
G(\Xi_t^{[m]}) - G(\Xi_0^{[m]}) = \frac{1}{2}\left\{ M(\Xi^{[m]})_t + M(\tilde{\Xi}^{[m]})_{T-t} - M(\tilde{\Xi}^{[m]})_T \right\}
$$

Lyons-Zheng decomposition

# 7. Condition **(B3)** – **(B5)**

Let $\mu^{\mathfrak{a}}_{Tail}$ be a regular conditional probability measure by the tail $\sigma$-field defined by

$$\mu^{\mathfrak{a}}_{Tail} = \mu(\cdot|\mathcal{T}(\mathfrak{M})(\mathfrak{a}).$$

Note that $\mu$ is quasi-Gibbs state implies $\mu^{\mathfrak{a}}_{Tail}$ is quasi-Gibbs state.

$\mu$ can be decomposed as $$\mu(\cdot) = \int_{\mathfrak{M}} \mu^{\mathfrak{a}}_{Tail}(\cdot)\mu(d\mathfrak{a}).$$

1. $\mu^{\mathfrak{a}}_{Tail}$ satisfies **(C1)**.
2. $\mu$ satisfies **(A1)–(A5)** $\Longrightarrow$ $\mu^{\mathfrak{a}}_{Tail}$ satisfies **(A1)–(A5)**, $\mu$-a.s. $\mathfrak{a}$.
3. $(\Xi_t, \mathbb{P}^{\mathfrak{a}}_{\mu^{\mathfrak{a}}})$ is reversible $\Longrightarrow$ **(C2)**
4. Lyons-Zheng decomposition $\Longrightarrow$ **(C3)**

$(\mathbf{X}, \mathbf{B})$ under $P^{\mathfrak{a}}_{\mathbf{x}}$ satisfies **(B3)-(B5)** for $\mu^{\mathfrak{a}}_{Tail} \circ \mathfrak{l}^{-1}$-a.s. $\mathbf{x}$.

# 8. Condition **(B2)**

## Assumption **(B2)**

For each $m \in \mathbb{N}$, (SDE-$m$) has a strong solution $\mathbf{Y}^m = F_{\mathbf{x}}^m(\mathbf{X}, \mathbf{B})$, and the pathwise uniqueness for solutions holds for each $m \in \mathbb{N}$.

$$
\begin{aligned}
& dY_t^{m,j} = dB_t^j + b_{\mathbf{X}}^m(t, (Y_t^{m,j}, \mathbf{Y}_t^{m,j\diamond}))dt, \\
& \mathbf{Y}_t^m \in \mathbf{S}_{\mathrm{sde}}(t, \mathbf{X}), \quad \text{for } \forall t \geq 0, \\
& \mathbf{Y}_0^m = \mathbf{x}^m = (x^1, x^2, \ldots, x^m),
\end{aligned} \qquad \text{(SDE-}m\text{)}
$$

Where $b_{\mathbf{X}}^m : [0, \infty) \times S^m \to (\mathbb{R}^d)^{\mathbb{N}}$ by

$$
b_{\mathbf{X}}^m(t, (u, \mathbf{v})) = b(u, \mathfrak{u}(\mathbf{v}) + \Xi_t^{m*}).
$$

# 8. Condition **(B2)** : Prelimiaries

Let $\mathbf{a} = \{a_k\}_{k\in\mathbb{N}}$ be a sequence of increasing sequence $a_k = \{a_k(r)\}_{r\in\mathbb{N}}$ such that $a_k(r) < a_{k+1}(r)$, $k, r \in \mathbb{N}$. We introduce a compact subset of $\mathfrak{M}$ defined by

$$\mathfrak{K}[a_k] = \{\xi \in \mathfrak{M} : \xi(S_r) \le a_k(r),\ \forall r \in \mathbb{N}\} \qquad \text{compact}$$

and put

$$\mathfrak{K} = \bigcup_{k\in\mathbb{N}} \mathfrak{K}[a_k].$$

We take a sequence such that $\mu(\mathfrak{K}) = 1$. If $\mu$ is translation invariant, we can take $a_k(r) = kr^d$.

We set $a_k^+(r) = a_k(r+1)$ and $\mathbf{a}^+ = \{a_k^+\}_{k\in\mathbb{N}}$

# 8. Condition **(B2)** : Prelimiaries

For $m \in \mathbb{N}$ let $\mathfrak{M}_{\mathrm{s.i.}}^{[m]} = \{(x, \xi) \in S^m \times \mathfrak{M} : \mathfrak{u}(x) + \xi \in \mathfrak{M}_{\mathrm{s.i.}}\}$

$$\mathfrak{H}[\mathbf{a}]_{p,q,k} = \Big\{ (x, \xi) \in \mathfrak{M}_{\mathrm{s.i.}}^{[m]} : x \in S_q^m, \xi \in \mathfrak{K}[a_k^+] \\ \inf_{j \neq k} |x^j - x^k| \geq 2^{-p}, \quad \inf_{j, y \in \mathrm{supp}\ \xi} |x^j - y| \geq 2^{-p} \Big\}$$

$$\mathfrak{H}[\mathbf{a}]_{q,k} = \bigcup_{p \in \mathbb{N}} \mathfrak{H}[\mathbf{a}]_{p,q,k} \quad \mathfrak{H}[\mathbf{a}]_k = \bigcup_{q \in \mathbb{N}} \mathfrak{H}[\mathbf{a}]_{q,k} \quad \mathfrak{H}[\mathbf{a}] = \bigcup_{k \in \mathbb{N}} \mathfrak{H}[\mathbf{a}]_k$$

## Facts

Suppose **(A1)** and **(A2)**

1. $\mathrm{Cap}^{\mu}(\mathfrak{K}[\mathbf{a}^+]^c) = 0$.

2. $\mathrm{Cap}^{\mu^{[m]}}(\mathfrak{H}[\mathbf{a}]^c) = 0$ for each $m \in \mathbb{N}$.

# 8. Condition **(B2)** : Preliminaries

Put $x = (x^1, x^2, \dots, x^m)$

$$b^m(x,\xi) = b(x^1, \sum_{j=2}^{m} \delta_{x^j} + \xi)$$

and let $\hat{b}^m(x,\xi)$ be a quasi-continuous version of $b^m(x,\xi)$. Put

$$\mathbf{N} = \{(p,q,k),(q,k),k : p,q,k \in \mathbb{N}\}.$$

Let $\{\mathfrak{I}_{\mathbf{n}}(\mathbf{m})\}_{\mathbf{m}\in\mathbb{N}}$ be an increase sequence of closed sets in $S^m \times \mathfrak{M}$ such that, for any $\mathbf{n} \in \mathbf{N}$,

$$\mathfrak{I}_{\mathbf{n}}(\mathbf{m}) \subset \mathfrak{I}_{\mathbf{n}}(\mathbf{m}+1) \quad \forall \mathbf{m} \in \mathbb{N}$$

and

$$\mathrm{Cap}^{\mu^{[m]}}((\bigcup_{\mathbf{m}\in\mathbb{N}} \mathfrak{I}_{\mathbf{n}}(\mathbf{m}))^c) = 0.$$

# 8. Condition **(B2)** : Prelimiaries

Let $c = c(\mathbf{m}, \mathbf{n})$ be constants such that $0 \le c \le \infty$ and that

$$c = \sup\left\{ \frac{|\hat{b}^m(x,\xi) - \hat{b}^m(x'.\xi)|}{|x - x'|} : (x,\xi) \sim_{\mathbf{n}} (x',\xi), x \neq x' \right.$$

$$\left. (x,\xi),(x',\xi) \in \mathfrak{H}[\mathbf{a}]_{\mathbf{n}} \cap \mathfrak{I}_{\mathbf{n}}(\mathbf{m}) \right\},$$

where $(x,\xi) \sim_{\mathbf{n}} (x',\xi)$ means $x$ and $x'$ are in the same connected component of $\{y \in S^{2m} : (y,\xi) \in \mathfrak{H}[\mathbf{a}]_{\mathbf{n}}\}$

**(E)** For each $m \in \mathbb{N}$ there exists a quasi-continuous version $\hat{b}^m$ of $b^m(x,\xi) = b(x^1, \sum_{j=2}^m + \xi)$ and $\{\mathfrak{I}_{\mathbf{n}}(\mathbf{m})\}$ such that $c(\mathbf{m},\mathbf{n}) < \infty$ for each $m \in \mathbb{N}$ and $\mathbf{n} \in \mathbb{N}$.

# 8. Condition **(B2)**

We put $\zeta_{\mathbf{m},\mathbf{n}}(\mathbf{X}^m, \Xi^{[m]}) = \inf\{t > 0 : (\mathbf{X}^m_t, \Xi^{[m]}_t) \notin \mathfrak{H}[\mathbf{a}]_{\mathbf{n}} \cap \mathfrak{I}_{\mathbf{n}}(\mathbf{m})\}$.

## Lemma 8.1

Assume **(A1)** – **(A5)**, **(E)**. Then **(B2)** holds.

## Mollifier (introduced in Osada 1996)

For each $m \in \mathbb{N}$, there exists a function $\chi_{\mathbf{n}}$, $\mathbf{n} \in \mathbf{N}$ satisfying the following conditions

1. $\chi_{\mathbf{n}}(x, \xi) = 0, \quad (x, \xi) \notin \mathfrak{H}[\mathbf{a}]_{\mathbf{n}+1}$.
2. $\chi_{\mathbf{n}}(x, \xi) = 1, \quad (x, \xi) \in \mathfrak{H}[\mathbf{a}]_{\mathbf{n}}$.
3. $0 \le \chi_{\mathbf{n}}(x, \xi) \le 1$, $|\nabla_x \chi_{\mathbf{n}}(x, \xi)|^2 \le \exists c$, $\mathbb{D}[\chi_{\mathbf{n}}, \chi_{\mathbf{n}}] \le \exists c'$.
4. $\chi_{\mathbf{n}} \in \mathcal{D}^{\mu^{[m]}}$.

# 8. Condition **(B2)**

Let $\quad \mathbf{J}^{[l]} = \{\mathbf{j} = (j_1, j_2, \dots, j_\ell)^m : 0 \le j_i \le \ell,\ \sum_{i=1}^{d} j_i = l\}$

Set $\quad \partial_{\mathbf{j}} = \dfrac{\partial^l}{\partial x_{\mathbf{j}}}$ for $\mathbf{j} \in \mathbf{J}^{[l]}$ $l \in \mathbb{N}$ $\quad \partial_{\mathbf{j}} = \mathit{identity}$ for $\mathbf{j} \in \mathbf{J}^{[0]}$.

For any $\ell \in \mathbb{N}$ we introduce the following

**(F1)** For each $\mathbf{j} \in \bigcup_{l=0}^{\ell} \mathbf{J}^{[l]}$, $\chi_{\mathbf{n}} \partial_{\mathbf{j}} b \in \mathcal{D}^{\mu^{[m]}}$ for all $\mathbf{n} \in \mathbf{N}$

**(F2)** For each $\mathbf{j} \in \mathbf{J}^{[\ell]}$, there exists $h_{\mathbf{j}} \in C(S^2 \setminus \{x, y\})$ such that

1. $\partial_{\mathbf{j}} b(x, \xi) = \sum_{k=1}^{m} \sum_{y \in \mathrm{supp}\ \xi} h_{\mathbf{j}}(x_k, y)$ for $(x, \xi) \in \mathfrak{H}[\mathbf{a}]$,
2. $\sup\{\sum_{k=1}^{m} \sum_{y \in \mathrm{supp}\ \xi} |h_{\mathbf{j}}(x_k, y)| : (x, \xi) \in \mathfrak{H}[\mathbf{a}]_{\mathbf{n}}\} < \infty$ for each $\mathbf{n} \in \mathbf{N}$.

## Lemma 8.2

Assume there exists $\ell \in \mathbb{N}$ such that **(F1)** and **(F2)** holds. Then **(E)** holds.

# (Examples)

We can check **(F1)** and **(F2)** for the following examples:

1. Lennard-Jones 6-12 potential  (**(F1)**,**(F2)** holds with $\ell = 1$.)
   $d = 3$ and $\Psi_{6,12}(x) = \{|x|^{-12} - |x|^{-6}\}$.

$$dX_t^j = dB_t^j + \frac{\beta}{2} \sum_{k \in \mathbb{N}, k \neq j} \{ \frac{12(X_t^j - X_t^j)}{|X_t^j - X_t^k|^{14}} - \frac{6(X_t^j - X_t^k)}{|X_t^j - X_t^k|^8} \} dt \quad (j \in \mathbb{N})$$

2. Riesz potentials  (**(F1)**,**(F2)** holds with $\ell = 1$.)
   $d < a \in \mathbb{N}$ and $\Psi_a(x) = (\beta/a)|x|^{-a}$.

$$dX_t^j = dB_t^j + \frac{\beta}{2} \sum_{k \in \mathbb{N}, k \neq j} \frac{X_t^j - X_t^k}{|X_t^j - X_t^k|^{a+2}} dt \quad (j \in \mathbb{N})$$

## (Examples)

logarithmic potential $\Psi(x, y) = \log|x-y|$

3. Sine RPF: $\beta = 1, 2, 4$. (**(F1)**,**(F2)** holds with $\ell = 1$.)

$$dX_t^j = dB_t^j + \lim_{L\to\infty}\left\{\frac{\beta}{2}\sum_{k\neq j,|X_t^k|<L}\frac{1}{X_t^j - X_t^k}\right\}dt \quad (j\in\mathbb{N})$$

4. Airy RPF : $\beta = 1, 2, 4$. (**(F1)**,**(F2)** holds with $\ell = 1$.)

$$dX_t^j = dB_t^j + \lim_{L\to\infty}\left\{\frac{\beta}{2}\sum_{k\neq j,|X_t^k|<L}\frac{1}{X_t^j - X_t^k} - \int_{|x|<L}\frac{\widehat{\rho}(x)dx}{-x}\right\}dt$$

$$(j\in\mathbb{N})$$

ここで $\widehat{\rho}(x) = \frac{1}{\pi}\sqrt{-x}\mathbf{1}(x<0)$ である.

# (Examples)

3.3 Ginibre RPF  (**(F1)**,**(F2)** holds with $\ell = 2$.)

$$dX_t^j = dB_t^j + \lim_{L\to\infty} \sum_{k\neq j,\ |X_t^j - X_t^k|<L} \frac{X_t^j - X_t^k}{|X_t^j - X_t^k|^2} dt \quad (j \in \mathbb{N})$$

$$dX_t^j = dB_t^j - X_t^j dt + \lim_{L\to\infty} \sum_{k\neq j,\ |X_t^k|<L} \frac{X_t^j - X_t^k}{|X_t^j - X_t^k|^2} dt \quad (j \in \mathbb{N})$$

# Relations of results

$\exists$**unique strong solution** $\Longleftarrow$ **(B1)** $\sim$ **(B5)** : **First tail theorem**

**(B1)** $\Longleftarrow$ **(A1)** $\sim$ **(A5)** : **Proposition 1 + Proposition 2**

**(B2)** $\Longleftarrow$ **(A1)** $\sim$ **(A5)** $+$ **(E)** : **Lemma 8.1**
$\Longleftarrow$ **(A1)** $\sim$ **(A5)** $+$ **(F1)**, **(F2)** : **Lemma 8.2**

**(B3)** $\sim$ **(B5)** $\Longleftarrow$ **(B2)** $+$ **(C1)** $\sim$ **(C3)** : **Second tail theorem**

## 9. Applications : preliminaries

Let $S_r = \{x \in \mathbb{R}^d : |x| \le r\}$ and

$$\mathfrak{M}_r^m = \{\xi \in \mathfrak{M} \,;\, \xi(S_r) = m\}, \quad r, m \in \mathbb{N}.$$

$\pi_r(\xi) = \xi(\cdot \cap S_r)$ and $\pi_r^c(\xi) = \xi(\cdot \cap S_r^c)$.

For $\xi \in \mathfrak{M}_r^m$, $\mathbf{x}_r^m(\xi) = (x_r^1(\xi), \dots, x_r^m(\xi)) \in S_r^m$ is called a $S_r^m$-coordinate of $\xi$, if $\pi_r(\xi) = \sum_{j=1}^m \delta_{x_r^j(\xi)}$.

For $f : \mathfrak{M} \to \mathbb{R}$, a function $f_{r,\cdot}^m(\cdot) : \mathfrak{M} \times S_r^m \to \mathbb{R}$ is called the $S_r^m$-representation for $f$ if $f_{r,\cdot}^m(\cdot)$ satisfies the following :

(1) $f_{r,\xi}^m$ is a permutation invariant function on $S_r^m$ for each $\xi \in \mathfrak{M}$.
(2) $f_{r,\xi(1)}^m(x) = f_{r,\xi(2)}^m(x)$ if $\pi_r^c(\xi(1)) = \pi_r^c(\xi(2))$, $\xi(1), \xi(2) \in \mathfrak{M}_r^m$.
(3) $f_{r,\xi}^m(\mathbf{x}_r^m(\xi)) = f(\xi)$ for $\xi \in \mathfrak{M}_r^m$.
(4) $f_{r,\xi}^m(\mathbf{x}) = 0$ for $\xi \notin \mathfrak{M}_r^m$.

Note that $f_{r,\cdot}^m(\cdot)$ is determined uniquely and $f(\xi) = \sum_{m=0}^\infty f_{r,\xi}^m(\mathbf{x}_r^m(\xi))$.

# 9. Applications : preliminaries

We set

$$\mathcal{B}_r(\mathfrak{M}) = \{f : \mathfrak{M} \to \mathbb{R} : \ f \text{ is } \sigma(\pi_r)\text{-measurable}\}$$

$$\mathcal{B}_\infty(\mathfrak{M}) = \bigcup_{r=1}^{\infty} \mathcal{B}_r(\mathfrak{M}), \ \text{the set of local functions}$$

When $f \in \mathcal{B}_r(\mathfrak{M})$, $S_r^m$-representation for $f$ is independent of $\xi$.
We introduce the set of local smooth functions defined by

$$\mathcal{D}_\infty^{\rm loc} = \{f \in \mathcal{B}_\infty(\mathfrak{M}) : \ f_{r,\xi}^m \text{ are smooth on } S_r^m \text{ for all } m, r \in \mathbb{N}, \xi \in \mathfrak{M} \ \},$$

Note that $\mathcal{D}_\infty^{\rm loc} \subset C(\mathfrak{M})$ and $\mathcal{D}_\infty^{\rm loc}$ is dense in $L^2(\mathfrak{M}, \mu)$, for a Borel measure $\mu$ on $\mathfrak{M}$.

## 9. Applications : Bilinear forms

For $f, g \in \mathcal{D}_{\infty}^{\mathrm{loc}}$、 $m, r \in \mathbb{N}$ we set

$$\mathbb{D}_r^m[f,g](\xi) = \frac{1}{2}\sum_i^m \nabla_{x_i} f_{r,\xi}^m(\mathbf{x}_r^m(\xi)) \cdot \nabla_{x_i} g_{r,\xi}^m(\mathbf{x}_r^m(\xi)),\ \xi \in \mathfrak{M}_r^m,$$

$$\mathbb{D}_r^m[f,g](\xi) = 0, \quad \xi \notin \mathfrak{M}_r^m,$$

Here $\xi = \sum_i \delta_{x_i}$, $\nabla_{x_i} = (\frac{\partial}{\partial x_{i1}}, , \dots, \frac{\partial}{\partial x_{id}})$.

Note that $\mathbb{D}_r^m[f,g]$ is independent of the choice of $S_r^m$-coordinate $\mathbf{x}_r^m(\xi)$ and well-defined. We put

$$\mathbb{D}_r[f,g] = \sum_{m=1}^{\infty} \mathbb{D}_r^m[f,g], \quad \xi \in \mathfrak{M}$$

# 9. Applications : Dirichlet forms

Let $\mu$ be a probability measure on $\mathfrak{M}$. We define bilinear forms on $\mathcal{D}_\infty^{\mathrm{loc}}$ :

$$\mathcal{E}_r^{\mu,m}(f,g) = \int_{\mathfrak{M}} \mathbb{D}_r^m[f,g](\xi)d\mu,$$
$$\mathcal{E}_r^{\mu}(f,g) = \int_{\mathfrak{M}} \mathbb{D}_r[f,g](\xi)d\mu.$$

Let $(\mathcal{E}^\mu, \mathcal{D}_o^\mu)$ be a bilinear form on $L^2(\mathfrak{M},\mu)$ with domain $\mathcal{D}_o^\mu$ defined by

$$\mathcal{D}_o^\mu = \{f \in \mathcal{D}_\infty^{\mathrm{loc}} \cap L^2(\mathfrak{M},\mu)\,;\, \mathcal{E}^\mu(f,f) < \infty\},$$
$$\mathcal{E}^\mu(f,f) \equiv \sup_{r\in\mathbb{N}} \mathcal{E}_r^\mu(f,f) = \lim_{r\to\infty} \mathcal{E}_r^\mu(f,f).$$

We make the following assumption:

$$(\mathcal{E}_r^{\mu,m}, \mathcal{D}_o^\mu) \text{ is closable on } L^2(\mathfrak{M},\mu) \text{ for each } m, r \in \mathbb{N} \qquad \text{(CL)}$$

# 9. Applications : Lemmas

Lemma 1 [Osada 1996, Lemmas 2.1-2.2, Theorem 2]

Under the assumption (CL)

(1) $(\mathcal{E}^{\mu}, \mathcal{D}_o^{\mu} \cap \mathcal{B}_r(\mathfrak{M}))$ and $(\mathcal{E}_r^{\mu}, \mathcal{D}_o^{\mu})$ are closable on $L^2(\mathfrak{M}, \mu)$ for each $r$.
Let $(\mathcal{E}^{upr}, \mathcal{D}_r^{upr})$ be the closure of $(\mathcal{E}^{\mu}, \mathcal{D}_o^{\mu} \cap \mathcal{B}_r(\mathfrak{M}))$,
and $(\mathcal{E}_r^{lwr}, \mathcal{D}_r^{lwr})$ be the closures of and $(\mathcal{E}_r^{\mu}, \mathcal{D}_o^{\mu})$

(2) $(\mathcal{E}^{\mu}, \mathcal{D}_o^{\mu})$ is closable on $L^2(\mathfrak{M}, \mu)$.
Let $(\mathcal{E}^{upr}, \mathcal{D}^{upr})$ be the closure of $(\mathcal{E}^{\mu}, \mathcal{D}_o^{\mu})$.

(3) $\{(\mathcal{E}_r^{lwr}, \mathcal{D}_r^{lwr})\}_{r \in \mathbb{N}}$ is increasing and $\{(\mathcal{E}^{upr}, \mathcal{D}_r^{upr})\}_{r \in \mathbb{N}}$ is decreasing.
Let $(\mathcal{E}^{lwr}, \mathcal{D}^{lwr})$ be the Dirichlet form

$$\mathcal{D}^{lwr} = \{f \in \bigcap_{r>0} \mathcal{D}_r^{lwr}; \lim_{r\to\infty} \mathcal{E}_r^{lwr}(f,f) < \infty\},$$
$$\mathcal{E}^{lwr}(f,f) = \lim_{r\to\infty} \mathcal{E}_r^{lwr}(f,f), \quad f \in \mathcal{D}^{lwr}.$$

# 9. Applications : Lemmas

Let $G_r^{lwr}(\alpha)$, $G^{lwr}(\alpha)$, $G_r^{upr}(\alpha)$ and $G^{upr}(\alpha)$ be resolvents of $(\mathcal{E}_r^{lwr}, \mathcal{D}_r^{lwr})$, $(\mathcal{E}^{lwr}, \mathcal{D}^{lwr})$, $(\mathcal{E}^{upr}, \mathcal{D}_r^{upr})$ and $(\mathcal{E}^{upr}, \mathcal{D}^{upr})$ on $L^2(\mathfrak{M}, \mu)$, respectively.

Lemma 2 [Osada 1996, Lemma 2.1.]

(1) $G_r^{lwr}(\alpha)$ converges to $G^{lwr}(\alpha)$ strongly in $L^2(\mathfrak{M}, \mu)$ for all $\alpha > 0$.

(2) $G_r^{upr}(\alpha)$ converges to $G^{upr}(\alpha)$ strongly in $L^2(\mathfrak{M}, \mu)$ for all $\alpha > 0$.

It is clear that $(\mathcal{E}^{upr}, \mathcal{D}^{upr}) \geq (\mathcal{E}^{lwr}, \mathcal{D}^{lwr})$, i.e.

$$\mathcal{D}^{upr} \subset \mathcal{D}^{lwr} \quad \text{and} \quad \mathcal{E}^{upr}(f, f) \geq \mathcal{E}^{lwr}(f, f), \quad f \in \mathcal{D}^{upr}.$$

**Problem**: When does the following hold?

$$(\mathcal{E}^{upr}, \mathcal{D}^{upr}) = (\mathcal{E}^{lwr}, \mathcal{D}^{lwr}).$$

This is not true in general. So we look for a sufficient condition.

# 9. Applications : About $(\mathcal{E}^{lwr}, \mathcal{D}^{lwr})$.

Remind that

1. Definition:

$$\mathcal{D}^{lwr} = \{f \in \bigcap_{r>0} \mathcal{D}_r^{lwr}; \lim_{r\to\infty} \mathcal{E}_r^{lwr}(f,f) < \infty\},$$

$$\mathcal{E}^{lwr}(f,f) = \lim_{r\to\infty} \mathcal{E}_r^{lwr}(f,f), \quad f \in \mathcal{D}^{lwr}.$$

2. $(\mathcal{E}^{lwr}, \mathcal{D}^{lwr})$ is closed on $L^2(\mathfrak{M}, \mu)$ and

$$G_r^{lwr}(\alpha) \to G^{lwr}(\alpha) \text{ strongly in } L^2(\mathfrak{M}, \mu),\ \forall \alpha > 0.$$

3. $(\mathcal{E}^{upr}, \mathcal{D}^{upr}) \geq (\mathcal{E}^{lwr}, \mathcal{D}^{lwr})$, that is,

$$\mathcal{D}^{upr} \subset \mathcal{D}^{lwr} \quad \text{and} \quad \mathcal{E}^{upr}(f,f) \geq \mathcal{E}^{lwr}(f,f), \quad f \in \mathcal{D}^{upr}.$$

# 9. Applications : Dirichlet form $(\mathcal{E}^+, \mathcal{D}^+)$.

Put

$$\mathcal{D}^+ = \Big\{ f \in L^2(\mathfrak{M}, \mu) : \exists f' \in L^2(\mathbb{R}^d \times \mathfrak{M}) \text{ s.t.}$$
$$- \int_{\mathbb{R}^d \times \mathfrak{M}} f(\delta_x + \eta)\{\mathbf{d}^\mu(x,\eta)\varphi(x,\eta) + \nabla_x \varphi(x,\eta)\} d\mu^{[1]}(x,\eta)$$
$$= \int_{\mathbb{R}^d \times \mathfrak{M}} f'(x,\eta)\varphi(x,\eta) d\mu^{[1]}(x,\eta),\ \forall \varphi \in C_c^\infty(\mathbb{R}^d) \otimes \mathcal{D}_\infty^{\mathrm{loc}} \Big\}.$$

Let us denote $f'$ by $D_x f$ and set

$$\mathcal{E}^+(f,g) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathfrak{M}} D_x f(x,\eta) \cdot D_x g(x,\eta) d\mu^{[1]}(x,\eta),\ f, g \in \mathcal{D}^+.$$

Under (A1), we see that $(\mathcal{E}^{lwr}, \mathcal{D}^{lwr}) = (\mathcal{E}^+, \mathcal{D}^+)$.
**Remark.** $(\mathcal{E}^+, \mathcal{D}^+)$ is not alway quasi regular.

# 9. Applications : Process associated with $(\mathcal{E}_r^{lwr}, \mathcal{D}_r^{lwr})$

$\mathbf{X}_t^r$ : the labeled process associated with the quasi-regular Dirichlet form $(\mathcal{E}^{lwr}, \mathcal{D}^{lwr})$.

$\mathbf{X}_t^r$ describe the system of interacting Brownian motions in which

1. each particle in $S_r$ at time 0 moves in $S_r$ and when it hits the boundary $\partial S_r$, it reflects and enters the domain $S_r$ immediately,
2. the particles out of $S_r$ stay the initial positions forever.

We denote by $\mu^{r,\xi}$ the regular conditional distribution defined by

$$\mu^{r,\xi}(\cdot) = \mu(\cdot|\sigma(\pi_r^c))(\xi) \quad \mu\text{-a.s. } \xi.$$

Remind that $\mu^{r,\xi}$ is a probability measure on

$$\mathfrak{M}_{r,\xi} = \{\eta \in \mathfrak{M} : \pi_r^c(\eta) = \pi_r^c(\xi)\}.$$

Then for $\mu$-a.s. $\xi$, the diffusion process $\mathbf{X}_t^r$ starting from $\xi$ is associated with the Dirichret form $(\mathcal{E}_r^{\mu^{r,\xi},lwr}, \mathcal{D}_r^{\mu^{r,\xi},lwr})$.

# 9. Applications : SDE for $\mathbf{X}^r$

We see that the system of infinite number of particles, in which only finite number of particles move, satisfies the following SDE: for $\mu$- a.s. $\xi$

$$dX_t^{r,j} = dB_t^j + \frac{1}{2}\mathbf{d}^{\mu}\left(X_t^{r,j}, \sum_{k\neq j}\delta_{X_t^{r,k}}\right)dt + \mathbf{n}^r(X_t^{r,j})dL_t^{r,j},\ 1 \le j \le \xi(S_r)$$

$$X_t^{r,j} = X_0^j, \quad j = \xi(S_r)+1, \xi(S_r)+2, \dots,$$

$$\mathbf{X}_0^r = \mathbf{x} \equiv (x_j)_{j\in\mathbb{N}},$$

where $L_t^{r,j}$, $j = 1, 2, \dots, \xi(S_r)$ are non-decreasing function satisfying

$$L_t^{r,j} = \int_0^t \mathbf{1}_{\partial S_r}(X_s^{r,j})dL_s^{r,j},$$

and $\mathbf{n}^r(x)$ is the inward normal unit vectors at $x \in \partial S_r$.

# 9. Applications : Assumption (A6)

**(A6)** There exist $\mathbf{b}_{s_1,s_2,\mathrm{p}} \in C_b(\mathbb{R}^d \times \mathfrak{M})$, $s_1, s_2, \mathrm{p} \in \mathbb{N}$, such that for $\mu$-a.s. $\xi$

$$\lim_{s_1\to\infty}\lim_{s_2\to\infty}\lim_{\mathrm{p}\to\infty}\sup_{r\geq s_1+s_2+1} \| \frac{1}{2}\mathbf{d}^{\mu} - \mathbf{b}_{s_1,s_2,\mathrm{p}} \|_{L^1(\mathbb{R}^d\times\mathfrak{M},\mu_R^{r,\xi,[1]})} = 0,$$

for each $R \in \mathbb{N}$, where $\mu^{r,\xi,[1]}$ is the 1-Campbell measure of $\mu^{r,\xi}$

$$\mu^{r,\xi,[1]}(dxd\eta) = 1_{S_r}(x)\rho^{r,\xi,1}(x)\mu_x^{r,\eta}(d\eta)dx, \quad (x,\eta) \in \mathbb{R}^d \times \mathfrak{M}.$$

and

$$\mu_R^{r,\xi,[1]}(dxd\eta)(\cdot) = \mu^{r,\xi,[1]}(dxd\eta)(\cdot \cap S_R \times \mathfrak{M})$$

## Theorem 3 [Kawamoto-Osada-T.]

Assume that **(A1)-(A6)** hold. Then $\{\mathbf{X}^r\}_{r\in\mathbb{N}}$ is tight in $W((\mathbb{R}^d)^{\mathbb{N}})$, and limit point $\mathbf{X} = (X^j)_{j\in\mathbb{N}}$ of $\{\mathbf{X}^r\}_{r\in\mathbb{N}}$ is a solution of the (ISDE1) :

$$dX_t^j = dB_t^j + \frac{1}{2}\mathbf{d}^{\mu}\left(X_t^j, \sum_{k:k\neq j}\delta_{X_t^k}\right)dt, \quad j \in \mathbb{N}.$$

with conditions **(C2)** and **(C3)**:

**(C2)** ($\mu$-AC) $\mu_t \prec \mu$ for all $t \geq 0$, where $\mu_t = P\left(\sum_{j\in\mathbb{N}}\delta_{X_t^j} \in \cdot\right)$.

**(C3)** (NBJ) for each $r, T \in \mathbb{N}$,

$$P\left(\sharp\{i \in \mathbb{N} : X_t^j \in S_r \text{ for some } t \in [0,T]\} < \infty\right) = 1.$$

## Uniqueness theorem

### Theorem [Osada-T.]

Assume that (A6) hold. Then solutions of the (ISDE1) :

$$dX_t^j = dB_t^j + \frac{1}{2}\mathsf{d}^{\mu}\Big(X_t^j, \sum_{k:k\neq j} \delta_{X_t^k}\Big)dt, \quad j \in \mathbb{N}.$$

with conditions **(C2)**, **(C3)** and **(B2)** are pathwise unique.

### Theorem 2 [Kawamoto-Osada-T.]

Assume that (A1)-(A6) and **(B2)** hold. Then

$$(\mathcal{E}^{upr}, \mathcal{D}^{upr}) = (\mathcal{E}^{lwr}, \mathcal{D}^{lwr}) = (\mathcal{E}^{+}, \mathcal{D}^{+}).$$

# (F1) and (F2)

Let $\mathbf{J}^{[l]} = \{\mathbf{j} = (j_1, j_2, \dots, j_\ell)^m : 0 \le j_i \le \ell, \ \sum_{i=1}^{d} j_i = l\}$

Set $\partial_{\mathbf{j}} = \dfrac{\partial^l}{\partial x_{\mathbf{j}}}$ for $\mathbf{j} \in \mathbf{J}^{[l]}$ $l \in \mathbb{N}$ $\partial_{\mathbf{j}} = \mathit{identity}$ for $\mathbf{j} \in \mathbf{J}^{[0]}$.

For any $\ell \in \mathbb{N}$ we introduce the following

**(F1)** For each $\mathbf{j} \in \bigcup_{l=0}^{\ell} \mathbf{J}^{[l]}$, $\chi_{\mathbf{n}} \partial_{\mathbf{j}} b \in \mathcal{D}^{\mu^{[m]}}$ for all $\mathbf{n} \in \mathbf{N}$

**(F2)** For each $\mathbf{j} \in \mathbf{J}^{[\ell]}$, there exists $h_{\mathbf{j}} \in C(S^2 \setminus \{x, y\})$ such that

1. $\partial_{\mathbf{j}} b(x, \xi) = \sum_{k=1}^{m} \sum_{y \in \mathrm{supp}\ \xi} h_{\mathbf{j}}(x_k, y)$ for $(x, \xi) \in \mathfrak{H}[\mathbf{a}]$,
2. $\sup\{\sum_{k=1}^{m} \sum_{y \in \mathrm{supp}\ \xi} |h_{\mathbf{j}}(x_k, y)| : (x, \xi) \in \mathfrak{H}[\mathbf{a}]_{\mathbf{n}}\} < \infty$ for each $\mathbf{n} \in \mathbf{N}$.

Thank you for your attention