Uniqueness of solutions of Infinite dimensional stochastic differential equations

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This talk is a review of the joint work[1] with Hirofumi Osada (Kyushu university).

Let S be a closed subset of $\mathbb{R}^d$ such that the interior S_{int} is a connected open set satisfying $\overline{S_{\text{int}}} = S$, and that the boundary ∂S has Lebesgue measure 0. Let S be the configuration space of unlabeled particles:

$$\mathsf{S} = \{\mathsf{s} = \sum_k \delta_{s_k}\ ;\ \ \mathsf{s}(K) < \infty, \text{for any compact } K \subset S\}.$$

S is a Polish space with the vague topology.

The configuration space of m labeled partcles is S^m, and that of infinite labeled particles is $S^{\mathbb{N}} = S^\infty$, respectively. For each $m \in \mathbb{N} \cup \{\infty\}$ we introduce the maps $\mathsf{u}_m : S^m \to \mathsf{S}$ and $\mathsf{u}_{[m]} : S^m \times \mathsf{S} \to \mathsf{S}$ defined by

$$\mathsf{u}_m(\mathbf{x}) = \sum_{k=1}^m \delta_{x_k}, \quad \mathsf{u}_{[m]}((\mathbf{x}, \mathsf{y})) = \mathsf{u}_m(\mathbf{x}) + \mathsf{y}, \qquad \mathbf{x} \in S^m,\ \mathsf{y} \in \mathsf{S}.$$

For $\boldsymbol{x} = (x^j)_{j\in\mathbb{N}} \in S^{\mathbb{N}}$, $m \in \mathbb{N}$ we put

$$\boldsymbol{x}^{m\diamond} = (x^1, \dots, x^{m-1}, x^{m+1}, \dots), \quad \boldsymbol{x}^{m\star} = (x^{m+1}, x^{m+2}, \dots) \in S^{\mathbb{N}},$$

and for $\mathbf{X} = (X^i)_{i\in\mathbf{N}} \in C([0,\infty), S^{\mathbb{N}})$ put

$$\mathsf{X}_t = \mathsf{u}_\infty(\mathbf{X}_t), \qquad \mathsf{X}_t^{m\diamond} = \mathsf{u}_\infty(\mathbf{X}_t^{m\diamond}), \quad \mathsf{X}_t^{m\star} = \mathsf{u}_\infty(\mathbf{X}_t^{m\star}).$$

For a suitable subset H of S, we put $\mathsf{H}^{[1]} = \mathsf{u}_{[1]}^{-1}(\mathsf{H}) \subset S \times \mathsf{S}$. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ be a probability space with filtration, and $\mathbf{B} = \{B^j\}_{j\in\mathbb{N}}$ be $\mathcal{F}_t$-adapted independent Brownian motions. For measurable functions

$$\sigma : \mathsf{H}^{[1]} \to \mathbb{R}^{d^2} \ \text{ and } \ b : \mathsf{H}^{[1]} \to \mathbb{R}^d,$$

we consider the following infinite dimensional stochastic differential equation:

$$\begin{aligned} &dX_t^j = \sigma(X_t^j, \mathsf{X}_t^{j\diamond})dB_t^j + b(X_t^j, \mathsf{X}_t^{j\diamond})dt, \\ &\mathsf{X} \in C([0,\infty), \mathsf{H}), \\ &\mathbf{X}_0 = \mathbf{s} \in \mathbf{H} \equiv \mathsf{u}_\infty^{-1}(\mathsf{H}). \end{aligned} \tag{ISDE}$$

In this talk we discuss the uniqueness of solutions of (ISDE).

Remark The coefficients σ and b are not always defined for any elemet $(x, \mathsf{s}) \in S \times \mathsf{S}$. The set H is chosen such that these coefficents are well defined and a solution has it as a state space.

We first make the following assumption.

(B.1) (ISDE) has a solution $(\mathbf{X}, \mathbf{B})$ on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$.

Suppose that $\boldsymbol{X}$ is a solution of (ISDE). For each $m \in \mathbb{N}$, $\sigma_{\mathsf{X}}^m : [0, \infty) \times S^m \to S^m$ and $b_{\mathsf{X}}^m : [0, \infty) \times S^m \to S^m$ are functions defined by

$$\sigma_{\mathsf{X}}^{m,j}(t, \boldsymbol{x}) = \sigma(x_j, \sum_{k \neq j}^{m} \delta_{x_k} + \mathsf{X}_t^{m\star}), \quad b_{\mathsf{X}}^{m,j}(t, \boldsymbol{x}) = b(x_j, \sum_{k \neq j}^{m} \delta_{x_k} + \mathsf{X}_t^{m\star}).$$

Then we consider the following system of finite dimensional SDEs:

$$\begin{aligned} &dY_t^{m,j} = \sigma_{\mathsf{X}}^{m,j}(t, \mathbf{Y}_t^m) dB_t^j + b^{m,j}(t, \mathbf{Y}_t^m) dt, \quad j = 1, \ldots, m, \\ &\mathsf{X} \in C([0, \infty), \mathsf{H}), \\ &(\mathbf{Y}_0^m, \boldsymbol{X}_0^{m\star}) = \mathbf{s}. \end{aligned} \tag{SDE-m}$$

We then make the following assumption. Set $\mathbf{B}^m = (B^j)_{j=1}^m$.

(B.2) $\forall m \in \mathbb{N}$, (SDE-m) has the unique strong solution $\mathbf{Y}^m = F_{\mathbf{s}}^m(\mathbf{B}^m, \mathbf{X}^{m\star})$.

We introduce the tail σ-fields on the path space $C([0, \infty), S^{\mathbb{N}})$:

$$\mathcal{T}_{\text{path}}(S^{\mathbb{N}}) = \bigcap_{m=1}^{\infty} \sigma[\mathbf{w}^{m\star}].$$

For a probability measure $\mathbb{P}$ on $C([0, \infty), S^{\mathbb{N}})$, we set

$$\mathcal{T}_{\text{path}}^{\{1\}}(S^{\mathbb{N}}; \mathbb{P}) = \{\mathbf{A} \in \mathcal{T}_{\text{path}}(S^{\mathbb{N}}) : \mathbb{P}(\mathbf{A}) = 1\}.$$

Let $\mathbb{P}_{\mathbf{s},\mathbf{B}}(\cdot) = P(\mathbf{X} \in \cdot | \mathbf{B})$ and P_{Br}^{∞} be the distribution of the standard Brownian motion on $(\mathbb{R}^d)^{\mathbb{N}}$ starting at the origin. We then make the following assumption.

(B.3) $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is $\mathbb{P}_{\mathbf{s},\mathbf{B}}$-trivial for P_{Br}^{∞}-a.s. $\mathbf{B}$.

(B,4) $\mathcal{T}_{\text{path}}^{\{1\}}(S^{\mathbb{N}}; \mathbb{P}_{\mathbf{s},\mathbf{B}})$ is independent of the distribution $P(\mathbf{X} \in \cdot)$ for P_{Br}^{∞}-a.s. $\mathbf{B}$.

The main theorem of this talk is the following.

Theorem([1])
(i) Assume that $(\boldsymbol{X}, \mathbf{B})$ satisfies **(B.1)-(B.3)**. Then $(\boldsymbol{X}, \mathbf{B})$ is a strong solution of (ISDE).
(ii) Assume that $(\boldsymbol{X}, \mathbf{B})$ satisfies **(B.1)-(B.4)**. Then $(\boldsymbol{X}, \mathbf{B})$ is the unique strong solution of (ISDE).

Some important examples satisfying **(B.1)-(B.4)** are given in the talk.

References

[1] Osada, H. and Tanemura, H. : Strong solutions of infinite dimensional stochastic differential equations and tail σ-fields, preprint arXiv:1412.8674(v7)[math.PR] .