Probability density function of SDEs with unbounded and path–dependent drift coefficient

Dai Taguchi

Osaka University

joint work with

Akihiro Tanaka (Osaka University / Sumitomo Mitsui Banking Corporation)

Stochastic Analysis and Related Topics
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Outline

Introduction

In this talk, we consider a path–dependent d–dimensional SDEs

$$\mathrm{d}X_t^x = b(t, X^x)\mathrm{d}t + \sigma(t, X_t^x)\mathrm{d}W_t,\ t \geq 0,\ X_0^x = x \in \mathbb{R}^d. \qquad (1)$$

- $W = (W_t)_{t\geq 0}$: d-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- drift coefficient $b : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \to \mathbb{R}^d$:
  - $\mathcal{B}([0, \infty)) \otimes \mathcal{B}(C([0, \infty); \mathbb{R}^d))/\mathcal{B}(\mathbb{R}^d)$-measurable
  - for each fixed $t > 0$, the map $C([0, \infty); \mathbb{R}^d) \ni w \mapsto b(t, w) \in \mathbb{R}^d$ is $\mathcal{B}_t(C([0, \infty); \mathbb{R}^d))/\mathcal{B}(\mathbb{R}^d)$-measurable.
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  Main tools are :
  - Analytical approach ⇝ Levi's parametrix method (PDE method)
  - Probabilistic approach ⇝ Malliavin calculus / Maruyama–Girsanov transform
- In this talk, we do not use Malliavin calculus, because we do not assume a smoothness for coefficients.

Known results (i)

- Assume that drift $b : \mathbb{R}^d \to \mathbb{R}^d$ is bdd, Hölder conti., and diffusion matrix σ is bdd, unif. elliptic and Hölder conti.

[1] Friedman, A. *Partial Differential Equations of Parabolic Type.* Dover Publications Inc.,

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- Then there exists the fundamental sol. of parabolic type PDE

$$(\partial_s + L)p(s,x;t,y) = 0, \quad \lim_{s \uparrow t} \int_{\mathbb{R}^d} f(y)p(s,x;t,y)\mathrm{d}y = f(x),$$

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$$p(s,x;t,y)$$

$$= g_{a(y)}(s,x;t,y) + \int_s^t \mathrm{d}u \int_{\mathbb{R}^d} \mathrm{d}z\, p(s,x;u,z)(L - L^y)g_{a(y)}(u,z;t,y),$$

  where $a := \sigma\sigma^\top$ and $g_{a(y)}(s,x;t,\cdot)$ is a pdf of "frozen" process $x + \sigma(y)W_{t-s}$ with generator L^y.

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  where $a := \sigma\sigma^\top$ and $g_{a(y)}(s, x; t, \cdot)$ is a pdf of “frozen" process $x + \sigma(y) W_{t-s}$ with generator L^y.
- Gaussian two-sided bound holds:

$$\widehat{C}_- g_{\widehat{c}_-(t-s)}(x, y) \leq p(s, x; t, y) \leq \widehat{C}_+ g_{\widehat{c}_+(t-s)}(x, y)$$

$$|\partial_{x_i} p(s, x; t, y)| \leq \frac{\widehat{C}_+}{(t - s)^{1/2}} g_{\widehat{c}_+(t-s)}(x, y).$$

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Known results (ii)

Maruyama[2] prove the following:

- Assume $b : \mathbb{R} \to \mathbb{R}$ is Lip. conti. and $\sigma = 1$.
- Then a pdf $p_t(x, y)$ of $X_t^x (= x + \int_0^t b(X_s^x) \mathrm{d}s + W_t)$ exists and has the following representation :

$$p_t(x, y)$$
$$= g_t(x, y) \mathbb{E}\left[\exp\left(\int_0^t b(x + W_s) \mathrm{d}W_s - \frac{1}{2} \int_0^t b(x + W_s)^2 \mathrm{d}s \right) \middle| x + W_t = y \right],$$

  where $g_t(x, \cdot)$ is a pdf of $x + W_t$
- This result is Girsanov theorem.

[2] Maruyama, G. *On the transition probability functions of the Markov process.* Nat. Sci. Rep. Ochanomizu Univ. **5**, 10–20, (1954).

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- Makhlouf show that pdf of a BM with random drift (L^p integrable) $\mathrm{d}X_t = b_t \mathrm{d}t + \mathrm{d}W_t$, $X_0 = x$ satisfies the following representation:

$$p_t(x,y) = g_t(x,y) + \int_0^t \mathbb{E}\left[\langle \nabla g_{t-s}(X_s,y), b_s\rangle\right] \mathrm{d}s, \text{ a.e., } y \in \mathbb{R}^d, \quad (2)$$

  This is the same representation for Levi's parametrix method. Indeed, if $b_s = b(X_s)$, then since $p_s(x,\cdot)$ is a pdf,

$$p_t(x,y) = g_t(x,y) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}z p_s(x,z)\langle \nabla g_{t-s}(z,y), b(z)\rangle.$$

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- Kusuoka show that if b is path-dept and bdd, and σ is bdd, UE, Hölder conti. then the Gaussian two-sided bound and the following representation holds:

$$p_t(x,y) = q(0,x;t,y) \mathbb{E}\left[Z_t(1, Y^{0,x}) \,\middle|\, Y_t^{0,x} = y \right], \text{ a.e., } y \in \mathbb{R}^d,$$

  where $Y_t^{s,x} = x + \int_s^t \sigma(r, Y_r^{s,x}) \mathrm{d}W_r$ with pdf $q(s,x;t,y)$ and

$$Z_t(q, Y^{0,x}) = \exp\left(\sum_{j=1}^d \int_0^t q(\sigma^{-1} b)_j (s, Y^{0,x}) \mathrm{d}W_s^j - \frac{1}{2} \int_0^t |q\sigma^{-1} b(s, Y^{0,x})|^2 \mathrm{d}s \right)$$

Goal :

Extend the results of Makhlouf and Kusuoka to SDEs with path-dept. and unbounded drift.

Existence and representations

Assumption 1

Suppose that the coefficients $b : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \to \mathbb{R}^d$ *and* $\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ *satisfy the following conditions:*

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(i) *The drift* b *is linear growth, that is, for each* $T > 0$*, there exists* $K(b,T) > 0$ *such that for any* $t \in [0,T]$ *and* $w \in C([0,T];\mathbb{R}^d)$*,*

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(ii) $a := \sigma\sigma^\top$ *is* α*-Hölder continuous in space and* $\alpha/2$*-Hölder continuous in time with* $\alpha \in (0, 1]$*, that is,*

$$\|a\|_\alpha := \sup_{t \in [0,\infty), x \neq y} \frac{|a(t, x) - a(t, y)|}{|x - y|^\alpha} + \sup_{x \in \mathbb{R}^d, t \neq s} \frac{|a(t, x) - a(s, x)|}{|t - s|^{\alpha/2}} < \infty.$$

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(iii) *The diffusion coefficient* σ *is bounded and uniformly elliptic, that is, there exist* $\underline{a}, \overline{a} > 0$ *such that for any* $(t,x,\xi) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$*,*

$$\underline{a}|\xi|^2 \le \langle a(t,x)\xi, \xi\rangle \le \overline{a}|\xi|^2.$$

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In particular, for any measurable functional $f : C([0, T]; \mathbb{R}^d) \to \mathbb{R}$ *such that the expectation* $\mathbb{E}[f(Y^{0,x})Z_T(1, Y^{0,x})]$ *exists, it holds that*

$$\mathbb{E}[f(X^x)] = \mathbb{E}[f(Y^{0,x})Z_T(1, Y^{0,x})], \tag{3}$$

where for $q \in \mathbb{R}$, $Z(q, Y^{0,x}) = (Z_t(q, Y^{0,x}))_{t\in[0,T]}$ *is a martingale defined by*

$$Z_t(q, Y^{0,x}) := \exp\left(\sum_{j=1}^{d} \int_0^t q\mu^j(s, Y^{0,x})\mathrm{d}W_s^j - \frac{1}{2}\int_0^t |q\mu(s, Y^{0,x})|^2 \mathrm{d}s\right),$$

$$\mu(t, w) := \sigma(t, w_t)^{-1} b(t, w),\ (t, w) \in [0, T] \times C([0, T]; \mathbb{R}^d).$$

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Moreover, for any $(t, x) \in (0, T] \times \mathbb{R}^d$, X_t^x *admits a pdf* $p_t(x, \cdot)$, *w.r.t Leb. meas. and it has the following representations: for a.e.* $y \in \mathbb{R}^d$,

$$\begin{aligned} p_t(x, y) &= q(0, x; t, y) + \int_0^t \mathbb{E}\left[\langle \nabla_x q(s, X_s^x; t, y), b(s, X^x) \rangle \right] \mathrm{d}s, \\ &= q(0, x; t, y) \mathbb{E}[Z_t(1, Y^{0,x}) \mid Y_t^{0,x} = y]. \end{aligned}$$

Remarks

Remark 1

For SDE $\mathrm{d}X_t = b(t,X)\mathrm{d}t + \mathrm{d}W_t$, *under linear growth condition on* b, *there exists a weak sol and uniqueness in law holds (see Corollary 3.5.16 in Karatzas and Shreve).*
In this case, $\sigma = I$, *that is if* $b = 0$ *then* $\mathrm{d}X_t = \mathrm{d}W_t$ *is important.*

[3] Olivera, C. and Tudor, C. A. *Density for solutions to stochastic differential equations with unbounded drift.* arXiv:1805.0671.

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Remark 2

Recently, Olivera and Tudor [3] *proved existence of pdf of* X_t^x *with Hölder continuous drift (unbounded),*
by using Malliavin calculus and Itô–Tanaka trick or Zvonkin transform,
that is, apply PDE $\lambda\phi_\lambda + L\phi_\lambda = b$*.*
Theorem 1 includes this results.

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Idea of proof Theorem 1:

- Check a "local" Novikov condition (see Corollary 3.5.14 in Karatzas and Shreve): for any fixed $T > 0$, there exist $n(T) \in \mathbb{N}$ and a sequence $\{t_0, \dots, t_{n(T)}\}$ such that $0 = t_0 < t_1 < \cdots < t_{n(T)} = T$ and

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_{n-1}}^{t_n} |q\mu(s, Y^{0,x})|^2 \mathrm{d}s\right)\right] < \infty, \text{ for all } n = 1, \dots, n(T).$$

Recall that a pdf $q(0, x; t, y)$ of $Y_t^{0,x}$ (without drift) satisfies the following GB:

$$\widehat{C}_- g_{\widehat{c}_-(t-s)}(x, y) \le q(s, x; t, y) \le \widehat{C}_+ g_{\widehat{c}_+(t-s)}(x, y), \tag{4}$$

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- By the def. of regular condi. prob., we have the second rep. for pdf.

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  Recall that a pdf $q(0, x; t, y)$ of $Y_t^{0,x}$ (without drift) satisfies the following GB:

$$\widehat{C}_- g_{\widehat{c}_-(t-s)}(x, y) \le q(s, x; t, y) \le \widehat{C}_+ g_{\widehat{c}_+(t-s)}(x, y), \qquad (4)$$

  Since $\mu = \sigma^{-1} b$ is of linear growth, so we can find a sequence $\{t_0, \dots, t_{n(T)}\}$. Thus $Z(1, Y^{0,x})$ is a martingale.
- Idea: $t_n - t_{n-1}$ is sufficiently small, then

$$\int_{\mathbb{R}^d} \exp(+c(t_n - t_{n-1})|y - x|^2) g_{c' t}(x, y) \mathrm{d}y < \infty.$$

- By the def. of regular condi. prob., we have the second rep. for pdf.
- Since $u(s, x; t) := \mathbb{E}[f(Y_t^{s,x})]$ is a solution of PDE

$$(\partial_s + L_s)u(s, x; t) = 0, \; u(t, x; t) = f(x), \; (s, x) \in [0, t) \times \mathbb{R}^d, \qquad (5)$$

  applying Itô's formula for $f(X_t^x)$, we obtain the first representation.

Gaussian two-sided bound and continuity

Now we consider the Gaussian two–sided bound and continuity for a pdf under the following *sub–linear growth condition* on the drift coefficient b.

Assumption 2

Suppose that for any $\delta, t > 0$, there exists $K_t(\delta) > 0$ such that $K_t(\delta)$ is increasing w.r.t t and for all $t > 0$ and $w \in C([0,t];\mathbb{R}^d)$,

$$|b(t,w)| \leq \delta \sup_{0\leq s\leq t} |w_s| + K_t(\delta).$$

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Remark 3

(i) *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function. If f is bounded on any compact subset of $\mathbb{R}^d$ and $|f(x)| = o(|x|)$ as $|x| \to \infty$, which is equivalent to the condition that for any $\delta > 0$, there exists a constant $K(\delta) > 0$ such that $|f(x)| \leq \delta|x| + K(\delta)$.*

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(ii) *If there exists $K > 0$ and $\beta \in (0, 1)$ such that*

$$|b(t, w)| \leq K(1 + |w_t^*|^\beta), \quad \text{for all } (t, w) \in [0, \infty) \times C([0, \infty); \mathbb{R}^d).$$

Then b satisfies Assumption 2 with $K_t(\delta) = K\{1 + (K/\delta)^{\beta/(1-\beta)}\}$.

Under sub–linear growth condition on b, we prove a Gaussian two–sided bound and a continuity for a pdf of X_t^x.

Theorem 2 (Taguchi and Tanaka 2018)

Suppose Assumption 1 and Assumption 2 hold.
Let $p_1, p_2, p_3 > 1$ *with* $p_1 \in (1, \frac{d}{d-1})$ *and* $1/p_1 + 1/p_2 + 1/p_3 = 1$.

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(i) *For each $(t, x) \in (0, T] \times \mathbb{R}^d$, the right hand side of the first representation of $p_t(x, y)$, is continuous with respect to y, that is, $p_t(x, \cdot)$ has a continuous version.*

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(ii) *There exist* $C_{\pm} \equiv C_{\pm}(p_1) > 0$ *such that for any* $(t, x) \in (0, T] \times \mathbb{R}^d$ *and a.e.* $y \in \mathbb{R}^d$, *it holds that*

$$p_t(x, y) \geq \frac{C_- g_{2^{-1}\widehat{c}_- t}(x, y)}{1 + \sup_{0 \leq s \leq t} \mathbb{E}\left[Z_s(1, Y^{0,x})^{-p_2}\right]^{1/p_2} \max_{i=1,2} \mathbb{E}\left[b(s, Y^{0,x})^{ip_3}\right]^{1/p_3}},$$

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and

$$p_t(x, y) \leq C_+ \left(1 + \sup_{0 \leq s \leq t} \mathbb{E}\left[Z_s(1, Y^{0,x})^{p_2}\right]^{1/p_2} \max_{i=1,2} \mathbb{E}\left[b(s, Y^{0,x})^{ip_3}\right]^{1/p_3}\right) g_{p_1 \widehat{c}_+ t}(x, y).$$

Under sub–linear growth cond. on b, moment of $Z_t(1, Y^{0,x})$ is finite.

Lemma 1

Suppose Assumption 1 and 2 hold. For any $r \in \mathbb{R}$, there exists $C > 0$ such that for all $(t, x) \in (0, T] \times \mathbb{R}^d$,

$$\sup_{0\le s\le t} \mathbb{E}[Z_s(1, Y^{0,x})^r] \le \begin{cases} 1, & \text{if } 2r^2 - r \le 0, \\ C\exp\left(CK(b,T)^2 t(1+|x|^2)\right), & \text{if } 2r^2 - r > 0,\ t \in (0, t_r], \\ C\exp\left(CK_T(\delta_{r,T})^2 t\right)\exp\left(\frac{|x|^2}{8\widehat{c}_+ T}\right), & \text{if } 2r^2 - r > 0,\ t \in (t_r, T], \end{cases}$$

where $t_r := \min\left\{T, \frac{1}{2K(b,T)\sqrt{3\underline{a}(2r^2-r)\widehat{c}_+}}\right\}$, $\delta_{r,t} := \frac{1}{2t\sqrt{3\widehat{c}_+\underline{a}(2r^2-r)}}$.

Idea of proof:
For $t \in (0, t_r]$, use liner growth.
For $t \in (t_r, T]$, use sub–linear growth with $\delta = \delta_{r,t}$.

Remark 4

If $d = 1, \sigma = 1$, $b(x) = x$, $rt \ge 2, rt^2 < \frac{1}{2}$ and then $\mathbb{E}[Z_t(1, Y^{0,x})^r] = \infty$.

Corollary 1

Let $p_t(x,\cdot)$ be a conti. version of a pdf of X_t^x . Then there exist $C_\pm > 0$ and $c_\pm > 0$ such that for any $t \in (0,T]$, $x, y \in \mathbb{R}^d$,

$$\frac{C_- g_{c_- t}(x,y)}{(1+|x|^2)\exp(c_+|x|^2)} \le p_t(x,y) \le C_+(1+|x|^2)\exp(c_+|x|^2) g_{c_+ t}(x,y).$$

Remark 5

Note that if b is bounded, then

$$\sup_{x\in\mathbb{R}^d} \sup_{0\le s\le t} \mathbb{E}\left[Z_s(1, Y^{0,x})^{\pm p_2}\right]^{1/p_2} < \infty,$$

thus we have the Gaussian two-sided

$$C_- g_{c_- t}(x,y) \le p_t(x,y) \le C_+ g_{c_+ t}(x,y).$$

Proof of Theorem 2

Gaussian two-sided bounds: The proof is based on Kusuoka's paper. Applying Fatou's lemma,

$$
\begin{aligned}
& q(0,x;t,y)\mathbb{E}\left[Z_t(1,Y^{0,x})^r \,\middle|\, Y_t^{0,x}=y\right] \\
& \leq q(0,x;t,y)\liminf_{s\to 0}\mathbb{E}\left[Z_{t-s}(1,Y^{0,x})^r \,\middle|\, Y_t^{0,x}=y\right] \\
& \leq q(0,x;t,y)\sup_{0\leq s<t}\mathbb{E}\left[Z_s(1,Y^{0,x})^r \,\middle|\, Y_t^{0,x}=y\right] \\
& = \sup_{0\leq s<t}\mathbb{E}\left[q(s,Y_s^{0,x};t,y)Z_s(1,Y^{0,x})^r\right] \text{ (by Markov property of } Y^{0,x}) \\
& \leq \widehat{C}_+ g_{\widehat{c}_+ t}(x,y) \text{ (by Itô's formula)} \\
& \quad + C_{r,p_1}\sup_{0\leq s\leq t}\mathbb{E}\left[Z_s(1,Y^{0,x})^{rp_2}\right]^{1/p_2}\max_{i=1,2}\mathbb{E}\left[b(s,Y^{0,x})^{ip_3}\right]^{1/p_3} g_{p_1\widehat{c}_+ t}(x,y)
\end{aligned} \tag{6}
$$

$$
\leq \frac{\widehat{C}_+ + C_{r,p_1}\sup_{0\leq s\leq t}\mathbb{E}\left[Z_s(1,Y^{0,x})^{rp_2}\right]^{1/p_2}\max_{i=1,2}\mathbb{E}\left[b(s,Y^{0,x})^{ip_3}\right]^{1/p_3}}{(2\widehat{c}_+ t)^{d/2}} < \infty, \tag{7}
$$

Hence (6) with $r=1$, we have the upper bound for $p_t(x,y)$.

(Lower bound) By using Schwarz's inequality, it holds that

$$\begin{aligned} 1 &= \mathbb{E}\left[Z_t(1, Y^{0,x})^{1/2} Z_t(1, Y^{0,x})^{-1/2} \,\middle|\, Y_t^{0,x} = y \right]^2 \\ &\leq \mathbb{E}\left[Z_t(1, Y^{0,x}) \,\middle|\, Y_t^{0,x} = y \right] \mathbb{E}\left[Z_t(1, Y^{0,x})^{-1} \,\middle|\, Y_t^{0,x} = y \right] \text{ a.e., } y \in \mathbb{R}^d, \end{aligned}$$

this implies

$$0 \leq \frac{1}{\mathbb{E}\left[Z_t(1, Y^{0,x})^{-1} \,\middle|\, Y_t^{0,x} = y \right]} \leq \mathbb{E}\left[Z_t(1, Y^{0,x}) \,\middle|\, Y_t^{0,x} = y \right], \text{ a.e., } y \in \mathbb{R}^d.$$

Therefore, from the representation of $p_t(x, y)$, we have

$$p_t(x, y) \geq \frac{q(0, x; t, y)^2}{q(0, x; t, y) \mathbb{E}\left[Z_t(1, Y^{0,x})^{-1} \,\middle|\, Y_t^{0,x} = y \right]} \geq 0, \text{ a.e., } y \in \mathbb{R}^d.$$

Applying (7) with $r = -1$, then $p_t(x, y)$ is estimated from below by

$$\begin{aligned} &\frac{(2\pi \widehat{c}_+ t)^{d/2} \frac{\widehat{C}_-^2 \exp\left(-\frac{|y-x|^2}{\widehat{c}_- t} \right)}{(2\pi \widehat{c}_- t)^d}}{\widehat{C}_+ + C_{-1,p_1} \sup_{0 \leq s \leq t} \mathbb{E}\left[Z_s(1, Y^{0,x})^{-p_2} \right]^{1/p_2} \max_{i=1,2} \mathbb{E}\left[b(s, Y^{0,x})^{ip_3} \right]^{1/p_3}} \\ &= \frac{\widehat{c}_+^{d/2} \widehat{c}_-^{-d/2} \widehat{C}_-^2 g_{2^{-1}\widehat{c}_-}(x, y)}{\widehat{C}_+ + C_{-1,p_1} \sup_{0 \leq s \leq t} \mathbb{E}\left[Z_s(1, Y^{0,x})^{-p_2} \right]^{1/p_2} \max_{i=1,2} \mathbb{E}\left[b(s, Y^{0,x})^{ip_3} \right]^{1/p_3}}. \end{aligned}$$

Sharp bounds for a pdf of Brownian motion with bounded drift

If the drift coefficient is bounded, then by using the parametrix method, we obtain the following representation on $p_t(x,y)$.

Theorem 3 (Taguchi and Tanaka 2018)

Let $\widetilde{X}^{s,x} = (\widetilde{X}^{s,x}_t)_{t \in [s,T]}$ be a solution of the following Markovian SDE

$$\widetilde{X}^{s,x}_t = x + \int_s^t \widetilde{b}(r, \widetilde{X}^{s,x}_r)\mathrm{d}r + \int_s^t \sigma(r, \widetilde{X}^{s,x}_r)\mathrm{d}W_r,$$

where $\widetilde{b} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded and measurable.
Suppose Assumption 1 holds and $b, \widetilde{b}$ are bounded. Then for any $(t,x,y) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d$, it holds that

$$p_t(x,y) = \widetilde{p}(0,x;t,y) + \int_0^t \mathbb{E}\left[\langle \nabla_x \widetilde{p}(s, X^x_s; t, y), b(s, X^x) - \widetilde{b}(s, X^x_s)\rangle\right] \mathrm{d}s,$$

where $\widetilde{p}(s,x;t,\cdot)$ is a pdf of $\widetilde{X}^{s,x}_t$.

Inspired by Qian and Zheng [4], we consider a sharp two–sided bound for a Brownian motion with path–dependent and bounded drift coefficient of the form

$$X_t^x = x + \int_0^t b(s, X^x)\mathrm{d}s + W_t,\ x \in \mathbb{R}^d,\ t \in [0, T], \tag{8}$$

by using Theorem 3 and bang–bang diffusion processes.
We define a d-dimensional bang–bang diffusion process with parameter $\alpha = (\alpha_1, \ldots, \alpha_d)^\top, \beta = (\beta_1, \ldots, \beta_d)^\top \in \mathbb{R}^d$:

$$Y_t^{x,\alpha,\beta} = x + \int_0^t \beta \operatorname{sgn}(\alpha - Y_s^{x,\alpha,\beta})\mathrm{d}s + W_t,$$

where $\beta\operatorname{sgn}(x) := (\beta_1\operatorname{sgn}(x_1), \ldots, \beta_d\operatorname{sgn}(x_d))^\top$, for each $x \in \mathbb{R}^d$.
Then it follows from Theorem 2 of Qian and Zheng that for any $t \in (0, T]$, $Y_t^{x,\alpha,\beta}$ admits a pdf, denoted by $q_t^{\alpha,\beta}(x, \cdot)$ which satisfies

$$q_t^{\alpha,\beta}(x, \alpha) = \prod_{i=1}^d \frac{2}{\sqrt{2\pi t}} \int_{|x_i - \alpha_i|/\sqrt{t}}^{\infty} z_i \exp\left(-\frac{(z_i - \beta_i\sqrt{t})^2}{2}\right)\mathrm{d}z_i.$$

[4] *Sharp bounds for transition probability densities of a class of diffusions.* C. R. Acad. Sci. Paris, Ser **335**(11), 953–957, (2002)

Corollary 2

Suppose Assumption 1 holds and the drift coefficient b is bounded. Then a pdf of a solution of (8)*, denoted by $p_t(x,\cdot)$ satisfies the following two–sided estimates: for any $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$q_t^{y,-\|b\|_\infty}(x,y) \le p_t(x,y) \le q_t^{y,\|b\|_\infty}(x,y).$$

Proof.

Let $x, y \in \mathbb{R}^d$ be fixed. Using Theorem 3 with $\widetilde{p} = q^{y,\pm\|b\|_\infty}$, we have

$$\begin{aligned} & p_t(x,y) - q_t^{y,\pm\|b\|_\infty}(x,y) \\ & = \int_0^t \mathbb{E}\left[\langle \nabla_x q_{t-s}^{y,\pm\|b\|_\infty}(X_s^x, y), b(s, X^x) - (\pm\|b\|_\infty)\mathrm{sgn}(y - X_s^x)\rangle\right] \mathrm{d}s. \end{aligned}$$

On the other hand, it holds that for any $s \in [0,t)$, $z \in \mathbb{R}^d$ and $w \in C([0,\infty);\mathbb{R}^d)$,

$$\begin{aligned} \partial_{z_i} q_{t-s}^{y,\|b\|_\infty}(z,y)(b^i(s,w) - \|b\|_\infty \mathrm{sgn}(y_i - z_i)) \le 0, \\ \partial_{z_i} q_{t-s}^{y,-\|b\|_\infty}(z,y)(b^i(s,w) + \|b\|_\infty \mathrm{sgn}(y_i - z_i)) \ge 0, \end{aligned}$$

thus we conclude the statement. □

Conclusions

- Under linear growth condi. of the drift b, provide two representations

$$p_t(x,y) = q(0,x;t,y) + \int_0^t \mathbb{E}\left[\langle \nabla_x q(s, X_s^x; t, y), b(s, X^x)\rangle\right] \mathrm{d}s, \text{ a.e., } y \in \mathbb{R}^d,$$
$$= q(0,x;t,y)\mathbb{E}[Z_t(1, Y^{0,x}) \mid Y_t^{0,x} = y], \text{ a.e., } y \in \mathbb{R}^d.$$

- Under sub–linear condi. on b, prove a Gaussian two sided bound:

$$\frac{C_- g_{c_- t}(x,y)}{(1+|x|^2)\exp(c_+|x|^2)} \le p_t(x,y) \le C_+(1+|x|^2)\exp(c_+|x|^2) g_{c_+ t}(x,y).$$

- Under bounded condi. on b, provide a representation

$$p_t(x,y) = \widetilde{p}(0,x;t,y) + \int_0^t \mathbb{E}\left[\langle \nabla_x \widetilde{p}(s, X_s^x; t, y), b(s, X^x) - \widetilde{b}(s, X_s^x)\rangle\right] \mathrm{d}s,$$

- Further results:
  - Hölder continuity of the map $y \mapsto p_t(x,y)$.
  - Application to numerical analysis for $\mathbb{E}[f(X_t^x)]$.
  - Existence of pdf of one-dim. SDEs with super–linear growth coefficients