Probability density function of SDEs with unbounded and path–dependent drift coefficient

Dai Taguchi (Osaka University)
joint work with
Akihiro Tanaka (Osaka University/Sumitomo Mitsui Banking Corporation)

Abstract

In this talk, we consider a path–dependent d–dimensional stochastic differential equations (SDEs) of the form

$$\mathrm{d}X_t^x = b(t, X^x)\mathrm{d}t + \sigma(t, X_t^x)\mathrm{d}W_t,\ t \geq 0,\ X_0^x = x \in \mathbb{R}^d, \tag{1}$$

where $W = (W_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The drift coefficient $b : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \to \mathbb{R}^d$ is a progressively measurable functional and diffusion matrix $\sigma : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a measurable function.

The existence and regularity of a probability density function (pdf) of X_t^x with respect to Lebesgue measure have been studied by many authors. It is well-known that by using Levi's parametrix method (see, [1]), if the drift $b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded Hölder continuous and diffusion matrix σ is bounded, uniformly elliptic and Hölder continuous, then there exists the fundamental solution of parabolic type partial differential equations (Kolmogorov equation) and it is a pdf of a solution of associated SDEs.

The existence of a pdf of a solution of SDEs with path–dependents and non-smooth coefficients have been studied recently. As a perturbation approach, the parametrix method and Maruyama–Girsanov theorem are useful tool in order to prove the existence of a pdf of a solution of SDEs with path–dependents coefficients. The case of bounded and path–dependent drift coefficient, Makhlouf [3] and Kusuoka [2] studied the existence, explicit representation, Gaussian (type) two–sided bound. In particular, Makhlouf showed that a pdf, denoted by $p_t(x, \cdot)$, of Brownian motion with random drift $\mathrm{d}X_t = b_t\mathrm{d}t + \mathrm{d}W_t$, $X_0 = x$ satisfies the following representation:

$$p_t(x, y) = g_t(x, y) + \int_0^t \mathbb{E}\left[\langle \nabla g_{t-s}(X_s, y), b_s\rangle\right] \mathrm{d}s,\ \text{a.e., } y \in \mathbb{R}^d, \tag{2}$$

where $g_t(x, y) := \frac{\exp\left(-|y-x|^2/2t\right)}{(2\pi t)^{d/2}}$, which is an analogue of the parametrix method. On the other hand, Kusuoka [2] showed that a pdf of a solution of path–dependent SDE (1) with bounded drift coefficient, denoted by $p_t(x, \cdot)$, has the following representation:

$$p_t(x, y) = q(0, x; t, y)\mathbb{E}\left[Z_t(1, Y^{0,x}) \mid Y_t^{0,x} = y\right],\ \text{a.e., } y \in \mathbb{R}^d, \tag{3}$$

where $q(0,x;t,\cdot)$ is the pdf of a solution of SDE without drift: $\mathrm{d}Y_t^{0,x}=\sigma(t,Y_t^{0,x})\mathrm{d}W_t$, $Y_0^{0,x}=x$, $Z_t(1,Y^{0,x})$ is the Girsanov density and $\mathbb{E}[\ \cdot \mid Y_t^{0,x}=y]$ is the expectation of a regular conditional probability given $Y_t^{0,x}=y$ for $y\in\mathbb{R}^d$. This representation is an analogue of Maruyama's result on the proof of Girsanov's theorem (see, [4]).

The aim of this paper is to show that the existence of a pdf and the representations (2) and (3) hold under linear growth condition on b. By using these representations, we will show that Gaussian type two–sided bound holds under sub–linear growth condition on b, that is, there exist positive constants $c_\pm$ and measurable functions $C_\pm:(0,T]\times\mathbb{R}^d\to(0,\infty)$ such that for any $(t,x)\in(0,T]\times\mathbb{R}^d$,

$$C_-(t,x)g_{c_-t}(x,y)\le p_t(x,y)\le C_+(t,x)g_{c_+t}(x,y),\ \text{a.e.},\ y\in\mathbb{R}^d.$$

Moreover, inspired by [5], we consider a sharp two–sided bound for a pdf of Brownian motion with path–dependent and bounded drift coefficient of the form $\mathrm{d}X_t^x=b(t,X^x)\mathrm{d}t+\mathrm{d}W_t$, by using bang-bang diffusion processes $\mathrm{d}Y_t^{x,\alpha,\beta}=\beta\mathrm{sgn}(\alpha-Y_t^{0,x,\alpha,\beta})\mathrm{d}t+dW_t$, $Y_0^{x,\alpha,\beta}=x$, with parameters $\alpha=(\alpha_1,\ldots,\alpha_d)^\top$, $\beta=(\beta_1,\ldots,\beta_d)^\top\in\mathbb{R}^d$. Here we define $\beta\mathrm{sgn}(x):=(\beta_1\mathrm{sgn}(x_1),\ldots,\beta_d\mathrm{sgn}(x_d))^\top$, for each $x\in\mathbb{R}^d$. More preciously, we will show that for any $(t,x,y)\in(0,T]\times\mathbb{R}^d\times\mathbb{R}^d$, it holds that

$$q_t^{y,-\|b\|_\infty}(x,y)\le p_t(x,y)\le q_t^{y,\|b\|_\infty}(x,y),$$

where $q_t^{\alpha,\beta}(x,\cdot)$ is a pdf of a bang-bang diffusion process $Y_t^{x,\alpha,\beta}$, which satisfies

$$q_t^{\alpha,\beta}(x,\alpha)=\prod_{i=1}^d\frac{2}{\sqrt{2\pi t}}\int_{|x_i-\alpha_i|/\sqrt{t}}^\infty z_i\exp\left(-\frac{(z_i-\beta_i\sqrt{t})^2}{2}\right)\mathrm{d}z_i.$$

References

[1] Friedman, A. *Partial Differential Equations of Parabolic Type.* Dover Publications Inc., (1964).

[2] Kusuoka, S. *Continuity and Gaussian two–sided bounds of the density functions of the solutions to path–dependent stochastic differential equations via perturbation.* Stochastic Process. Appl., **127**, 359–384, (2017).

[3] Makhlouf, A. *Existence and Gaussian bounds for the density of Brownian motion with random drift.* Commun. Stoch. Anal. **10**(2), 151–162, (2016).

[4] Maruyama, G. *On the transition probability functions of the Markov process.* Nat. Sci. Rep. Ochanomizu Univ. **5**, 10–20, (1954).

[5] Qian, Z. and Zheng, W. *Sharp bounds for transition probability densities of a class of diffusions.* C. R. Acad. Sci. Paris, Ser **335**(11), 953–957, (2002).