NON-CONVERGENCE OF EQUILIBRIUM MEASURES AT ZERO TEMPERATURE LIMIT

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1. Abstruct

We study behaviour of equilibrium measures at zero temperature limit in the context of symbolic dynamics. A fundamental problem in this context is whether a sequence of equilibrium measures converges or not. The answer significantly depends on the dimension of symbolic dynamics and the regularity of a potential.

We start with basic definitions. Let $\mathcal{A}$ be a finite set and $d \geq 1$ a positive integer. Consider the space $\mathcal{A}^{\mathbb{Z}^d}$ endowed with the product topology of the discrete topology. We define $\mathbb{Z}^d$ action σ on $\mathcal{A}^{\mathbb{Z}^d}$ by

$$\sigma^i(x)_j = x_{i+j}$$

for every $i, j \in \mathbb{Z}^d$. We call the pair $(\mathcal{A}^d, \sigma)$ the d-dimensional *full shift* over $\mathcal{A}$. A Borel probability measure μ on $\mathcal{A}^{\mathbb{Z}^d}$ is σ*-invariant* if $\mu(\sigma^i B) = \mu(B)$ for every $i \in \mathbb{Z}^d$ and Borel sets B. Denote by $\mathcal{M}_\sigma$ the space of σ-invariant Borel probability measures endowed with the weak*-topology. For a continuous function ϕ on $\mathcal{A}^{\mathbb{Z}^d}$ define the *pressure* by

$$\mathcal{P}(\phi) = \sup\left\{ h_\mu + \int \phi \, d\mu : \mu \in \mathcal{M}_\sigma \right\}$$

where h_μ is the Kolmogorov entropy. A σ-invariant measure attaining the supremum is called an *equilibrium measure* for ϕ. For every continuous function there exists an equilibrium measure.

In the case of $d = 1$ the uniqueness of equilibrium measure holds for every Hölder continuous function [Bo]. However this is not true when $d \geq 2$ even for locally constant functions [K]. For $n \geq 1$ define the *d-dimensional box* Λ_n with size n by $\Lambda_n = \{-n+1, \ldots, 0, \ldots, n-1\}^d$. For $x \in \mathcal{A}^{\mathbb{Z}^d}$ denote by x_{Λ_n} the restriction of x to Λ_n. Recall that a function ϕ on $\mathcal{A}^{\mathbb{Z}^d}$ is *locally constant* if $\phi(x) = \phi(y)$ whenever $x_{\Lambda_n} = y_{\Lambda_n}$ for some $n \geq 1$.

Take a continuous function ϕ on $\mathcal{A}^{\mathbb{Z}^d}$. For $\beta \in \mathbb{R}$ denote by μ_β an equilibrium measure for $\beta\phi$. We consider the following question:

- Does a sequence $\{\mu_\beta\}$ of equilibrium measures converge when β goes to infinity?

The parameter β is called *inverse temperature* and the limit $\lim_{\beta\to\infty} \mu_\beta$ is called the *zero temperature limit*, if it exists. Remark that we consider convergence with regard to continuous parameter families, since there always exists a convergent subsequence because of the compactness of $\mathcal{M}_\sigma$.

In the case of $d = 1$, the zero temperature limit exists for every locally constant function [Br, CGU, L]. On the other hand, there exists a Lipschitz continuous function for which the limit does not exist [CH]. Remark that we have a unique choice of sequences of equilibrium measures in the above cases. In the case of $d \geq 2$ the situation is more complicated. We prove the following.

Theorem 1. [CS] *There exists a locally constant function on a two-dimensional full shift for which every sequence of equilibrium measures does not converge.*

In contrast to one-dimensional case, there may be several choices of sequences and we emphasis that our result is non-convergence for every choice of sequences. For $d \geq 3$ the same non-convergent result for locally constant function is proved by Chazottes and Hochman[CH].

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