Random walks, Laplacians, and volumes in sub-Riemannian geometry

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Sub-Riemannian geometry

A (constant-rank) sub-Riemannian manifold is a triple $(M, \blacktriangle, g)$ where M is a smooth (connected) n-dimensional manifold, $\blacktriangle$ is a smoothly-varying, rank-k $(2 \leq k < n)$, bracket-generating distribution, and g is a smoothly-varying Riemannian metric on $\blacktriangle$.

Locally, $\blacktriangle$ and g can be specified by giving a smooth orthonormal frame $X_1, \ldots, X_k$.

Example: The Heisenberg group. Let $X = \partial_x - (y/2)\partial_z$ and $Y = \partial_y + (x/2)\partial_z$ be orthonormal in $\mathbb{R}^3$. Then $[X, Y] = \partial_z = Z$ (the Reeb vector field, thinking of this as a contact structure). Also, X, Y, and Z are left-invariant.

Sub-Riemannian volumes and Laplacians

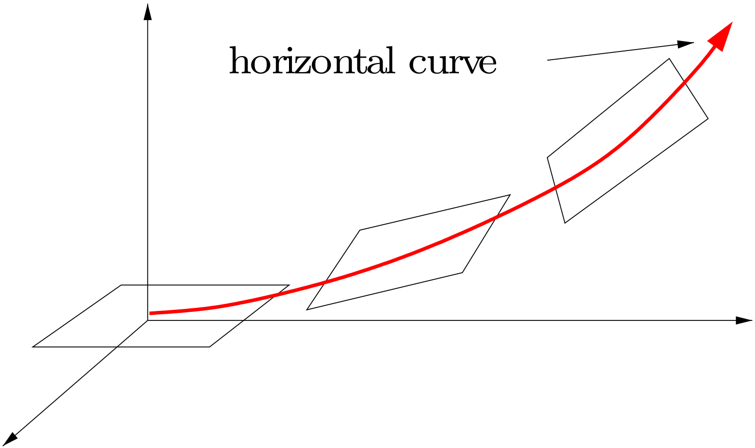
On a sR manifold, there is a natural notion of horizontal gradient. Then an obvious approach to defining a Laplacian is by $\Delta_\omega = \mathrm{div}_\omega \circ \mathrm{grad}_H$, for some “sub-Riemannian volume” ω. (This gives a self-adjoint operator.) Unfortunately, there is no canonical choice of ω, in general.

- Popp volume (for equiregular structure) defined by Montgomery [2001] (already defined by Brockett in some special cases in 1981).
- spherical Hausdorff volume (or Hausdorff volume).
- volumes coming from Lie group structure or “nice” Riemmanian extensions, in some cases.

In general Popp and spherical Hausdorff do not coincide unless the “tangent space to the sub-Riemannian manifold” (in the Gromov-Hausdorff sense) it is independent of the point , and they are not smooth one w.r.t. one another (C^3 but not C^5 in contact sR geometry [Agrachev, Barilari, Boscain, Gauthier 2012-2014]).

Geodesics

Horizontal curves have tangent vectors in ▲. They have lengths, and one can ask to minimize the length between two points. This gives rise to a distance, making M into a metric space. Length minimizers are geodesics or/and abnormals (which we'll ignore).



Co-tangent cylinder

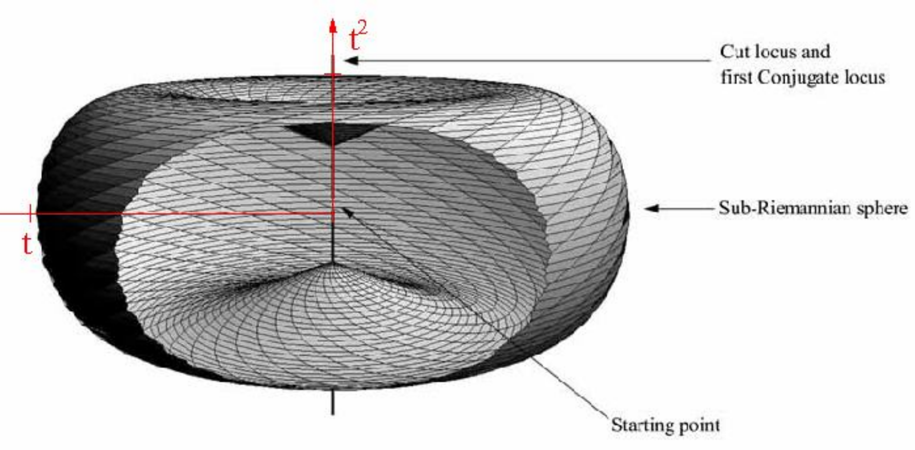
The geodesics are given by projections on the manifold of solutions of the Hamiltonian system having as Hamiltonian

$$H(q,p) = \frac{1}{2}\sum_{i}^{k}\langle p, X_i(q)\rangle^2 \quad \lambda = (q,p) \in T^*M, \text{ and } X_i \text{ a local o.n. frame.}$$

Arclength parameterized geodesics belong to $H = \frac{1}{2}$, which will be a non-compact cylinder. For Heisenberg, at the origin, it is $\{p_x^2 + p_y^2 = 1\}$.

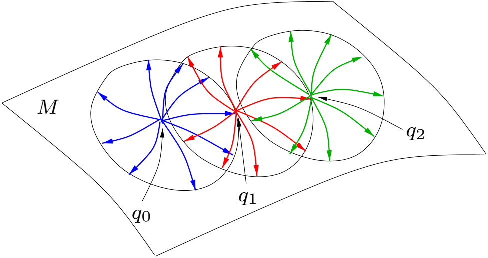
The Heisenberg geodesics

Let $\partial_x - (y/2)\partial_z$ and $\partial_y + (x/2)\partial_z$ be orthonormal in $\mathbb{R}^3$:



Isotropic random walk: The Riemannian case

The Laplacian is the infinitesimal generator of $\sqrt{2}$-BM, the limit of an isotropic random walk q_t^ε, with step size ε in time $\varepsilon^2/(2n)$.



That is, for any $\varphi \in C_0^\infty(M)$, with μ_q uniform probability measure,

$$\begin{aligned}
\Delta\varphi(q) &= \lim_{\varepsilon\to 0} \frac{2n}{\varepsilon^2} \left(\int_{\mathbb{S}^{n-1}} \varphi\left(\exp_q(\varepsilon,\theta)\right)\, d\mu_q(\theta) - \varphi(q) \right) \\
&= \lim_{\varepsilon\to 0} \frac{2n}{\varepsilon^2} \left(\mathbb{E}\left[\varphi\left(q^\varepsilon_{\varepsilon^2/(2n)} \right) \middle| \, q_0^\varepsilon = q \right] - \varphi(q) \right) \\
&:= \lim_{\varepsilon\to 0} L^{\varepsilon^2/(2n)} \varphi(q).
\end{aligned}$$

Probabilistic aside: convergence of random walks

Previous andom walk/flight/etc. approximations to diffusions in geometry:

- Pinsky (1976): a random flight/isotropic transport process in Riemannian case, via semi-group/resolvent methods
- Gordina-Laetsch (2017): same thing for sR case
- Lebeau-Michel (2010 and 2015): convergence also of spectrum of some types of random walks in Riemannian and sR cases
- Breuillard-Friz-Huesmann (2009): convergence of Euclidean random walks in rough path topology, can be viewed as a case of sR convergence
- Angst-Bailleul-Tardif (2015): "Kinetic BM" on Riemannian manifold, C^1 curves with diffusing velocity
- Xue-Mei Li (2016): a vertical diffusion on $SO(n)$; similar to previous
- Ishiwata-Kawabi-Kotani-Namba (2017-) CLTs for random walks on graphs embedded into Euclidean or sR manifolds
- von Renesse (2004) and Kuwada (2012) use random walk approximations to rigorously construct coupled Brownian motions on (time-dependent) Riemannian manifolds

Pathspace

Following Stroock-Varadhan (1979), let Ω_M be the space of continuous paths from $[0,\infty)$ to M. For $\omega \in \Omega_M$, let ω_t be the position of ω at time t. Then we can define a metric on Ω by

$$d_{\Omega_M}(\omega, \tilde{\omega}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\sup_{0\leq t\leq i} d_M(\omega_t, \tilde{\omega}_t)}{1 + \sup_{0\leq t\leq i} d_M(\omega_t, \tilde{\omega}_t)}.$$

This metric makes Ω_M into a Polish space (convergence is uniform convergence on bounded time intervals) with Borel σ-algebra and natural filtration. We will generally be interested in a (Markov) family of probability measures, indexed by points of M, written as P_q for $q \in M$.

Controlled interpolation

Let P_q^h be a family of random walks from q with "steps" at time $0, h, 2h, 3h, \ldots$, namely continuous, piecewise deterministic processes, Markov at step-times.

We also need a "controlled interpolation" condition: for any $\rho > 0$, any compact $Q \subset M$, and any $\alpha > 0$, there exists $h_0 > 0$ such that

$$\frac{1}{h} P_q^h \left[\sup_{0 \le s \le h} d_M \left(q_0^h, q_s^h \right) \le \rho \right] > 1 - \alpha$$

whenever $q \in Q$ and $h < h_0$.

Automatically satisfied if step-size goes uniformly to 0, as before.

Convergence in general

Theorem (Boscain-N.-Rizzi 2017)

Let M be a (sub)-Riemannian manifold (possibly rank-varying) with a smooth diffusion operator L. Further, suppose that the diffusion generated by L, which we call q^0, does not explode, and let P_q be the corresponding probability measure on Ω_M starting from q. Similarly, let P_q^h be the probability measures on Ω_M corresponding to a sequence of random walks q_t^h as above with $q_0^h = q$, and let L_h be the associated operators. Suppose that, for any $\varphi \in C_0^\infty(M)$, we have that

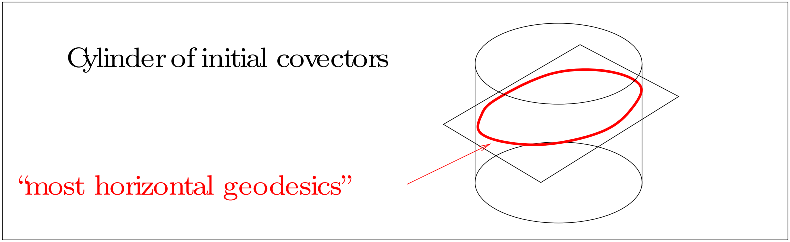
$$L_h\varphi \to L\varphi \quad \text{uniformly on compacts as } h \to 0,$$

and also suppose that the controlled interpolation condition holds for the q_t^h. Then if $q_h \to q$ as $h \to 0$, we have that $P_{q_h}^h \to P_q$ as $h \to 0$.

A kind of Donsker invariance…

sR geodesic random walk

Geodesics are well understood, so we might try to use them to determine a Laplacian. But there's no uniform probability measure on the cylinder of "unit" co-vectors. We could take the "most horizontal geodesics"…



…but this is meaningless in general, because there is no canonical splitting of either the tangent or co-tangent spaces.

Walk/Laplacian w.r.t. a splitting

Let $\mathbf{c}$ be smooth choice of splitting/vertical complement such that $T_qM = \blacktriangle_q \oplus \mathbf{c}_q$. This equivalently gives horizontal subspace of T^*M_q. The sR metric g does induce a uniform measure on the resulting $(k-1)$-dim. sphere of horizontal "unit" co-vectors. So we can construct a corresponding horizontal random walk, which converges to a diffusion generated by a "Laplacian" $L^{\mathbf{c}}$. (Or more directly define horizontal divergence.)

We can also let the probability measure on co-vectors have an independent vertical component with some appropriate polynomial decay– this doesn't affect the limiting process or operator.

Heisenberg/3D contact

For the Heisenberg group, at the origin, let

$$v_1 = dx + a\,dz \quad \text{and} \quad v_2 = dy + b\,dz$$

give an o.n. basis for the space of horizontal co-vectors, for some real a and b. We wish to choose a (unit) horizontal covector uniformly at random, which means a covector

$$\cos\theta\, v_1 + \sin\theta\, v_2 = \cos\theta\, dx + \sin\theta\, dy + \left(a\cos\theta + b\sin\theta\right)\, dz$$

where θ is chosen uniformly at random from $[0, 2\pi)$.

The associated ransom walk, in the parabolic scaling limit, acts on smooth functions by

$$L\varphi = \varphi_{xx} + \varphi_{yy} + b\,\varphi_x + a\,\varphi_y.$$

So there's a 1-1 correspondence between splittings and first-order terms here.

Compatibility

Still no canonical choice in general, but we can ask about compatibility: When does $L^{\mathbf{c}} = \Delta_\omega$?

Theorem (Gordina-Laetsch 2016 via Riemannian extension, Grong-Thalmaier 2016 some cases, Boscain-N.-Rizzi 2017)

For any complement $\mathbf{c}$ *and volume* ω*,* Δ_ω *and* $L^{\mathbf{c}}$ *have the same principal symbol. Moreover* $L^{\mathbf{c}} = \Delta_\omega$ *if and only if*

$$L^{\mathbf{c}} - \Delta_\omega = \sum_{i=1}^{k} \sum_{j=k+1}^{n} c_{ji}^{j} X_i + \mathrm{grad}_H(\theta) = 0,$$

where $\theta = \log|\omega(X_1, \ldots, X_n)|$ *and* c_{ij}^{ℓ} *are the structural functions associated with an orthonormal frame* $X_1, \ldots, X_k$ *for* $\blacktriangle$ *and a frame* $X_{k+1}, \ldots, X_n$ *for* $\mathbf{c}$*.*

This is our coordinate-free version of the theorem; it is suitable for broad classes of examples…

Contact structures

Theorem (Boscain-N.-Rizzi 2017)

Let $(M, \blacktriangle, g)$ be a contact sub-Riemannian structure. For any ω there exists a unique $\mathbf{c}$ such that $L^{\mathbf{c}} = \Delta_\omega$. In this case $\mathbf{c} = \mathrm{span}\{X_0\}$, with

$$X_0 = Z - J^{-1}\,\mathrm{grad}_H(\theta), \qquad \theta = \log|\omega(X_1, \dots, X_k, Z)|, \tag{1}$$

where Z is the Reeb vector field and $J : \blacktriangle \to \blacktriangle$ is the contact endomorphism.

Contact structures have a natural Riemannian extension, obtained by declaring the Reeb vector field a unit vector orthonormal to $\blacktriangle$. It turns out that the Riemannian volume of this extension is Popp volume.

Theorem (Boscain-N.-Rizzi 2017)

Let $\mathcal{P}$ be the Popp volume. The unique complement $\mathbf{c}$ such that $L^{\mathbf{c}} = \Delta_{\mathcal{P}}$ is generated by the Reeb vector field. Moreover, $\mathcal{P}$ is the unique volume (up to constant rescaling) with this property.

Continued

The inverse problem, namely for a fixed $\mathbf{c}$, to find a volume ω such that $L^{\mathbf{c}} = \Delta_\omega$ is a more complicated (and in general has no solution). In the contact case, we gave explicitly a necessary and sufficient condition.

For Carnot groups, Popp volume and (left) Haar volume are proportional. There exists a complement with $L^{\mathbf{c}} = \Delta_{\mathcal{P}}$, but it is not unique in general.

More on the canonicalness of Popp volume

We say a volume on a (sub)-Riemannian manifold is *N-intrinsic* if (informally) it depends only on the nilpotent approximation at each point. For a Riemannian manifold, such a volume is the standard one, up to a constant. Popp volume and spherical Hausdorff volume are both N-intrinsic.

A (sub)-Riemannian structure is *equi-nilpotentizable* if the nilpotent approximations at any two points are isometric.

If a (sub)-Riemannian manifold is equi-nilpotentizable, the only N-intrinsic volume, up to a constant, is Popp volume.

Lack of complement

Consider the quasi-contact structure on $M = \mathbb{R}^4$ with coordinates (x, y, z, w), and $g(z)$ any strictly increasing, positive function, given by the global o.n. frame

$$X = \frac{1}{\sqrt{g}}\left(\partial_x + \frac{1}{2}y\partial_w\right), \quad Y = \frac{1}{\sqrt{g}}\left(\partial_y - \frac{1}{2}x\partial_w\right), \quad Z = \frac{1}{\sqrt{g}}\partial_z.$$

This structure is equi-nilpotentizable, so the the Popp volume

$$\mathcal{P} = \frac{g^{5/2}}{\sqrt{2}} dx \wedge dy \wedge dz \wedge dw$$

is the unique (up to constant) N-intrinsic volume.

Lemma (Boscain-N.-Rizzi 2017)

For this M, there is no complement $\mathbf{c}$ *such that* $L^{\mathbf{c}} = \Delta_{\mathcal{P}}$.

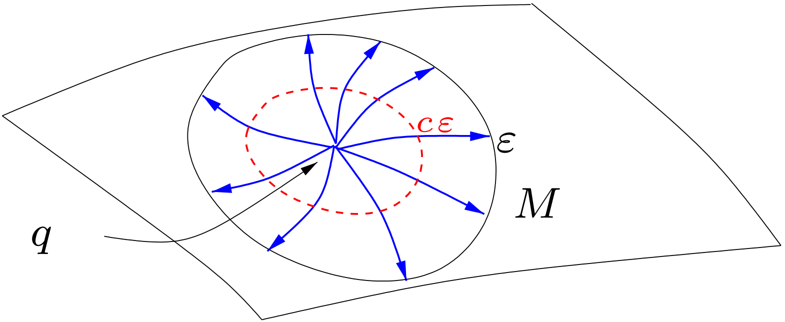
Non-standard volumes in Riemannian case

Let M be a Riemannian manifold, with Riemannian volume $\mathcal{R}$. If $\omega = e^h \mathcal{R}$, with $h \in C^\infty(M)$, then

$$\Delta_\omega = \Delta_{\mathcal{R}} + \mathrm{grad}(h).$$

If we want a random walk that gives this operator in the limit, we should take the volume into account in a way the previous isotropic walk does not.

Volume-sampled walk



We pull back the $n-1$ form $i_{\dot\gamma(t)}\omega := \omega_{\gamma(t)}(\dot\gamma(t),\ldots)$, defined along the geodesic γ, through the exponential map to obtain a probability measure on $\mathbb{S}^{n-1}$ given by

$$\mu_q^{c\varepsilon} = \frac{1}{N(q,c\varepsilon)}\left(\exp_q(c\varepsilon,\cdot)^* i_{\dot\gamma(c\varepsilon)}\omega\right)_q, \text{ where } c\in[0,1].$$

The volume-sampled operator

Then we choose the geodesic for the next step of the random walk by $\mu_q^{c\varepsilon}$. For $c = 0$, we get the same thing as before. But in general, in the limit we get the diffusion associated to the operator

$$L_\omega^c \varphi(q) = \lim_{\varepsilon \to 0} \frac{2n}{\varepsilon^2} \left(\int_{\mathbb{S}^{n-1}} \varphi\left(\exp_q(\varepsilon, \theta)\right) \, d\mu_q^{c\varepsilon}(\theta) - \varphi(q) \right).$$

Lemma (Agrachev-Boscain-N.-Rizzi, in press)

With the notation above,

$$L_\omega^c = \Delta_\omega + (2c-1)\,\mathrm{grad}(h).$$

Hence $L_\omega^c = \Delta_\omega$ (and L_ω^c with domain $C_c^\infty(M)$ is essentially self-adjoint on $L^2(M, \omega)$) if and only if either ω is proportional to $\mathcal{R}$ or $c = 1/2$.

Significance

This points out the special role played by the Riemannian volume. It's also interesting that $c = 1$ doesn't give the right answer (we haven't seen this written down before, but apparently it is somewhat familiar to experts, for example, Bismut).

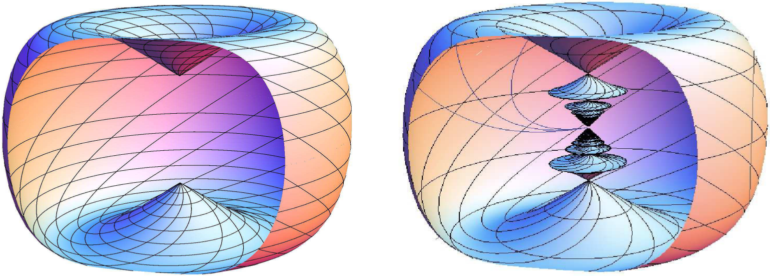
However $c = 1/2$ is equivalent to weighting by the "total volume" seen along the geodesic, which after the fact seems natural.

That this absolutely continuous change of measure on $\mathbb{S}^{n-1}$, from μ_q to $\mu_q^{c\varepsilon}$, results in a drift can be nicely interpreted as a "discretization" of Girsanov's theorem.

Volume-sampling in the sR context

Here, many issues arise:

- Do we include geodesics past their cut points?



- If so, do we use the induced signed measure, or take absolute value?

sR context

- The computations are difficult, in part because the cylinder of initial co-vectors is not compact.

For Heisenberg, we get results like in the Riemannian case. For higher dimensional Carnot groups, the principal symbol won't be "right," in general. We can say some things, but it's a complicated situation and hard to go beyond groups.