LOCAL ESTIMATES OF ITERATED PARAPRODUCTS

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1. Introduction

Many singular SPDEs have motivations from statistical physics, quantum field theory, etc., but they are sometimes ill-posed without "renormalizations". The theory of *paracontrolled calculus* by Gubinelli, Imkeller and Perkowski [4] made it possible to show the local well-posedness results for such renormalized SPDEs. Compared with the famous theory of *regularity structures* by Hairer [6], the PC theory has an advantage in showing detailed properties [5, 8, 9, 1, 3, 7] (global well-posedness, ergodicity, etc.) but it is not algebraically sophisticated. Our ultimate goal is to show the equivalence of RS and PC and construct a new theory which has both advantages of RS and PC.

One of the main differences between the two theories is in the definition of solutions. In PC, solutions are written by using the Bony's paraproduct [2]. In RS, solutions are described based on local estimates. Therefore in order to get the relationship between these concepts, we need local estimates of Bony's paraproduct.

This talk is based on a joint work with Ismaël Bailleul.

2. Main results

Our main theorems are the local estimates of iterated paraproducts and the Hopf algebra structure behind these estimates.

We consider the Bony's paraproducts on the Euclidean space $\mathbb{R}^d$. Let $(\Delta_i)_{i\ge -1}$ be the Littlewood-Paley blocks. For any distribution f on $\mathbb{R}^d$, we write

$$f_i = \Delta_i f, \quad f_{i-} = \sum_{j\le i-2} f_j.$$

Then the Bony's paraproduct is defined by

$$f \otimes g = \sum_i f_{i-} g_i.$$

For any $f^i \in \mathcal{S}'(\mathbb{R}^d)$, $i = 1, \dots, n$, we define the *iterated paraproduct*

$$(f^1, \dots, f^n) = \sum_i (f^1, \dots, f^n)_i, \quad (f^1, \dots, f^n)_i = (f^1, \dots, f^{n-1})_{i-} (f^n)_i$$

Note that

$$(f^1, \dots, f^n) \neq (\dots((f^1 \otimes f^2) \otimes f^3) \dots \otimes f^{n-1}) \otimes f^n.$$

We conjecture that they have similar local estimates.

Theorem 2.1. *Let $\alpha_1, \dots, \alpha_n > 0$.*

(1) *Let* $f^1 \in C^{\alpha_1}$. *Define*

$$\Delta_{yx} f^1 = f^1(y) - \sum_{|k|<\alpha_1} \frac{(y-x)^k}{k!} \partial^k f^1(x).$$

Then one has $|\Delta_{yx} f^1| \lesssim |y-x|^{\alpha_1}$.

(2) *Let* $f^1 \in C^{\alpha_1}$ *and* $f^2 \in C^{\alpha_2}$. *Define*

$$\Delta_{yx}(f^1, f^2) = (f^1, f^2)(y) - \sum_{|k|<\alpha_1+\alpha_2} \frac{(y-x)^k}{k!} \partial_*^k (f^1, f^2)(x) - \sum_{|l|<\alpha_1} \frac{(y-x)^l}{l!} \partial^l f^1(x) \Delta_{yx} f^2,$$

where

$$\partial_*^k (f^1, f^2) = \partial^k (f^1, f^2) - \sum_{k=l+m, |l|<\alpha_1, |m|\geq\alpha_2} \frac{k!}{l!m!} (\partial^l f^1)(\partial^m f^2)$$

Then one has $|\Delta_{yx}(f^1, f^2)| \lesssim |y-x|^{\alpha_1+\alpha_2}$.

(3) *Let* $f^i \in C^{\alpha_i}$, $i = 1, \dots, n$. *Define*

$$\Delta_{yx}(f^1, \dots, f^n) = (f^1, \dots, f^n)(y) - \sum_{|k|<\alpha_1+\cdots+\alpha_n} \frac{(y-x)^k}{k!} \partial_*^k (f^1, \dots, f^n)(x) - \sum_{m=1}^{n-1} \sum_{|l|<\alpha_1+\cdots+\alpha_m} \frac{(y-x)^l}{l!} \partial_*^l (f^1, \dots, f^m)(x) \Delta_{yx}(f^{m+1}, \dots, f^n)$$

with some coefficients $\partial_*^l (f^1, \dots, f^m)(x)$ *defined continuously from* $f^1, \dots, f^n$. *Then one has* $|\Delta_{yx}(f^1, \dots, f^n)| \lesssim |y-x|^{\alpha_1+\cdots+\alpha_n}$.

Next we define the Hopf algebra which describes these "extended Taylor series". Let $W = \bigcup_{k\in\mathbb{N}} \{1, \dots, n\}^k$ and let $\mathcal{W}$ be the commutative algebra freely generated by W. We define the *coproduct*

$$\mathring{\Delta}(i_1 \dots i_k) = (i_1 \dots i_k) \otimes \mathbf{1} + \mathbf{1} \otimes (i_1 \dots i_k) + \sum_{l=1}^{k-1} (i_{l+1} \dots i_k) \otimes (i_1 \dots i_l).$$

Then $\mathcal{W}$ is a Hopf algebra. Moreover, let $\tilde{W} = W \times \mathbb{N}^d$ and define the differential map $\partial_i : \tilde{W} \to \tilde{W}$ by $\partial_i \tau_m = \tau_{m+e_i}$. We extend the coproduct $\mathring{\Delta}$ by

$$\mathring{\Delta} \partial_i = (\partial_i \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_i) \mathring{\Delta}.$$

We have independent symbols $\{\mathbf{X}_i\}_{i=1}^d$ and define

$$\mathring{\Delta} \mathbf{X}_i = \mathbf{X}_i \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{X}_i.$$

Let $\mathcal{H}$ be the commutative algebra freely generated by $\tilde{W} \cup \{\mathbf{X}_i\}_{i=1}^d$. Then $(\mathcal{H}, \mathring{\Delta})$ is a Hopf algebra. We next define the new coproduct Δ by

$$\Delta = \exp(\sum_{i=1}^d \mathbf{X}_i \otimes \partial_i) \mathring{\Delta}.$$

Here we assume $\partial_i \mathbf{X}_j = 0$ and denote by $\mathbf{X}_i$ the map $\tau \mapsto \mathbf{X}_i \tau$. Then $(\mathcal{H}, \Delta)$ is again a Hopf algebra.

Fix $\beta_1, \dots, \beta_n > 0$. We define the homogeneity $|\cdot|$ on $\mathcal{H}$ by

$$|(i_1 \dots i_k)_m| = \beta_{i_1} + \cdots + \beta_{i_k} - |m|, \quad |\mathbf{X}_i| = 1.$$

Here $|m| = \sum_{i=1}^d m_i$. Now $\mathcal{H}$ is *graded* but contains negative homogeneities. Hence we focus only on the algebra

$$\mathcal{H}_+ = \mathcal{H}/\langle \tau\sigma \in \mathcal{H}; |\sigma| < 0 \rangle.$$

Then $\mathcal{H}^+$ forms the regularity structure $(\mathcal{H}_+, G)$ with the character group G on $\mathcal{H}^+$. The following theorem is the another form of the above theorem.

Theorem 2.2. *Let $\alpha_1, \dots, \alpha_n > 0$.*

(1) *Given $f^i \in C^{\alpha_i}$, $i = 1, \dots, n$, we define $(\mathsf{\Pi}, \mathsf{g})$ by*

$$\mathsf{\Pi}(i_1 \dots i_k)_m(x) = \mathsf{g}_x((i_1 \dots i_k)_m) = \partial_*^m (f^{i_1}, \dots, f^{i_k})(x).$$

Then $(\mathsf{\Pi}, \mathsf{g})$ is a model on $(\mathcal{H}_+, G)$.

(2) *Let $\beta > 0$ and $g \in C^\beta$. We define the $\mathcal{H}_+$-valued function*

$$\begin{aligned} \boldsymbol{g} = &\sum_{|k| < \beta + \alpha_1 + \cdots + \alpha_n} \partial_*^k (g, f^1, \dots, f^n) \frac{\mathbf{X}^k}{k!} \\ &+ \sum_{i=1}^{n-1} \sum_{|k_i| < \beta + \alpha_1 + \cdots + \alpha_i} \partial_*^{k_i} (g, f^1, \dots, f^i) \frac{\mathbf{X}^{k_i}}{k_i!} ((i+1) \dots n) \\ &+ \sum_{|m| < \beta + \alpha_1 + \cdots + \alpha_n} (\partial^m g) \frac{\mathbf{X}^m}{m!} (1 \dots n). \end{aligned}$$

Then one has $\boldsymbol{g} \in \mathcal{D}^{\beta + \alpha_1 + \cdots + \alpha_n}$.

References

[1] S. Albeverio and S. Kusuoka, *The invariant measure and the flow associated to the Φ^4_3-quantum field model*, arXiv:1711.07108.

[2] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. École Norm. Sup. (4) **14** (1981), no. 2, 209-246.

[3] M. Gubinelli and M. Hofmanová, *Global solutions to elliptic and parabolic Φ^4 models in Euclidean space*, arXiv:1804.11253.

[4] M. Gubinelli, P. Imkeller, and N. Perkowski, *Paracontrolled distributions and singular PDEs*, Forum Math. Pi **3** (2015), e6, 75pp.

[5] M. Gubinelli and N. Perkowski, *KPZ reloaded*, Comm. Math. Phys. **349** (2017), no. 1, 165-269.

[6] M. Hairer, *A theory of regularity structures*, Invent. Math. **198** (2014), no. 2, 269-504.

[7] M. Hoshino, *Global well-posedness of complex Ginzburg-Landau equation with a space-time white noise*, arXiv:1704.04396.

[8] J.-C. Mourrat and H. Weber, *The dynamic Φ^4_3 model comes down from infinity*, Comm. Math. Phys. **356** (2017), no. 3, 673-753.

[9] P. Tsatsoulis and H. Weber, *Spectral gap for the stochastic quantization equation on the 2-dimensional torus*, Ann. Inst. Henri Poincaré Probab. Stat. **54** (2018), no. 3, 1204-1249.

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