Central limit theorem for random walks on nilpotent covering graphs with weak asymmetry

Hiroshi KAWABI (Keio University)

(jointwork with Satoshi ISHIWATA (Yamagata)
& Ryuya NAMBA (Okayama))

arXiv:1808.08856

Workshop on Stochastic Analysis and Related Topics

November 21, 2018 @ Okayama

Introduction

RWs on Γ-nilpotent covering graphs

♠ Γ : finitely generated torsion free nilpotent group of step r

- torsion free: If $\gamma^n = 1_\Gamma$, then $n = 0$ (and $\gamma = 1_\Gamma$).
- nilpotent: There exists some $r \in \mathbb{N}$ such that

$$\Gamma \supset [\Gamma, \Gamma] \supset \cdots \supset \Gamma^{(r)}(:= [\Gamma, \Gamma^{(r-1)}]) = \{1_\Gamma\}$$

♠ A nilpotent covering graph X is a covering graph of a finite graph X_0 whose covering transformation group is Γ.

In other words, Γ acts on X freely and the quotient graph $X_0 = \Gamma \backslash X$ is finite.

♠ X is called a crystal lattice if Γ is abelian ($r = 1$).

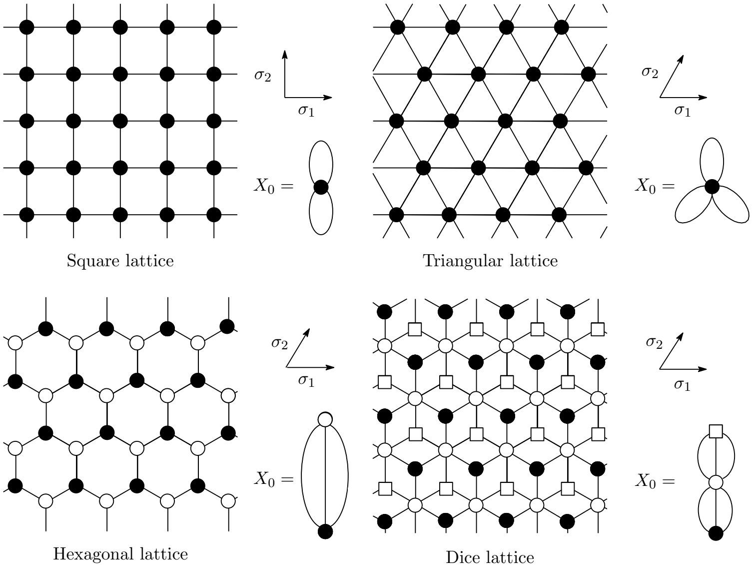


Figure : Crystal lattices with $\boldsymbol{\Gamma = \langle \sigma_1, \sigma_2 \rangle \cong \mathbb{Z}^2}$

3D disctrete Heisenberg group : $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_1^{-1}, \gamma_2^{-1}, \gamma_3^{-1} \rangle$

$$\gamma_1\gamma_3 = \gamma_3\gamma_1,\ \ \gamma_2\gamma_3 = \gamma_3\gamma_2,\ \ [\gamma_1, \gamma_2](= \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}) = \gamma_3$$

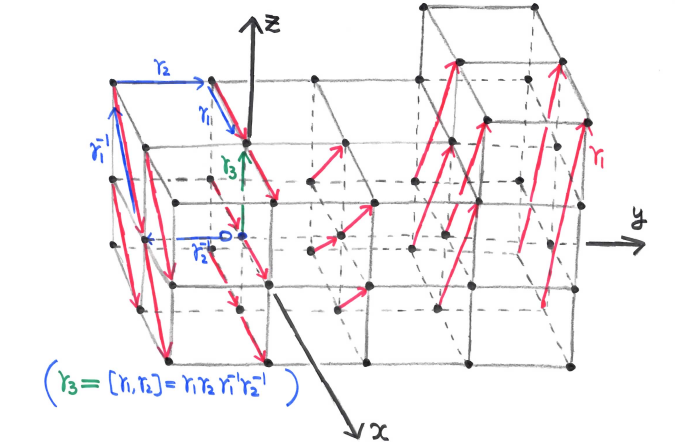


Figure : A part of the Cayley graph of $\Gamma = \mathbb{H}_3(\mathbb{Z})$

- For an edge $e \in E$, the origin, the terminus and the inverse edge of e are denoted by $o(e), t(e)$ and $\overline{e}$, respectively.
- $E_x := \{e \in E \mid o(e) = x\}\ (x \in V)$.

♠ A RW on X is characterized by giving the transition probability $p : E \longrightarrow [0,1]$ satisfying the Γ-invariance,

$$p(e) + p(\overline{e}) > 0, (e \in E), \quad \& \quad \sum_{e \in E_x} p(e) = 1,\ (x \in V).$$

$\Longrightarrow$ This induces a time homogeneous Markov chain

$$(\Omega_x(X), \mathbb{P}_x, \{w_n\}_{n=0}^{\infty}),$$

where $\Omega_x(X)$ stands for the set of all paths in X starting at x.

♠ By Γ-invariance of p, we may consider the RW $(\Omega_{\pi(x)}(X_0), \mathbb{P}_{\pi(x)}, \{w_n\}_{n=0}^{\infty})$ ($\pi : X \to X_0$: covering map).

- $Lf(x) := \sum_{e \in E_x} p(e) f\big(t(e)\big)$: transition operator.
- n-step transition probability; $p(n, x, y) := L^n \delta_y(x)$.

Assumption (Irreducibility)

The Markov chain $\{w_n\}_{n=0}^{\infty}$ on X_0 is irreducible, that is,
$\forall x, y \in V_0$, $\exists n = n(x, y) \in \mathbb{N}$ s.t. $p(n, x, y) > 0$.

$\boxed{\text{Remark}}$ RW on X : irreducible $\overset{\Longrightarrow}{\not\Longleftarrow}$ RW on X_0 : irreducible.

♠ By the Perron-Frobenius theorem,

$$\exists!\, m : V_0 \longrightarrow (0, 1] : L\text{-invariant measure, s.t.}$$

$$\sum_{x \in V_0} m(x) = 1 \quad \& \quad {}^t Lm(x) = m(x) \quad (x \in V_0).$$

♠ We also write $m : V \longrightarrow (0, 1]$ for the lift of m to X.

- $\widetilde{m}(e) := p(e) m\big(o(e)\big)$ (the conductance of $e \in E$).

♠ We define the homological direction γ_p of the RW by

$$\gamma_p := \sum_{e \in E_0} \widetilde{m}(e)e \in \mathrm{H}_1(X_0, \mathbb{R}).$$

♠ RW: $(m\text{-})$symmetric $\overset{\text{def}}{\Longleftrightarrow} \widetilde{m}(e) = \widetilde{m}(\overline{e}) \overset{\text{iff}}{\Longleftrightarrow} \gamma_p = 0$.

Our Problem

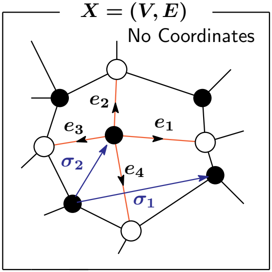
Functional CLT (Donsker type invariance principle)

♡ Abelian case: Ishiwata–K–Kotani ('17, JFA)

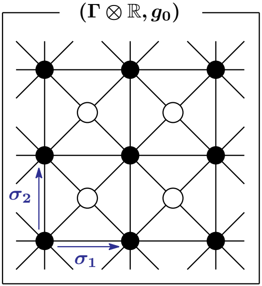
$$\Big(\frac{\Phi_0(w_{[nt]}) - [nt]\rho_{\mathbb{R}}(\gamma_p)}{\sqrt{n}}\Big)_{t \geq 0} \underset{n \to \infty}{\Longrightarrow} (B_t)_{t \geq 0}$$, where

$\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{R}) \twoheadrightarrow \Gamma \otimes \mathbb{R} (\cong \mathbb{Z}^d \otimes \mathbb{R} = \mathbb{R}^d)$,
$\Phi_0 : X \to (\Gamma \otimes \mathbb{R}, g_0)$ is the "standard realization", and
(B_t): standard BM on $\Gamma \otimes \mathbb{R}$ with Albanese metric g_0.

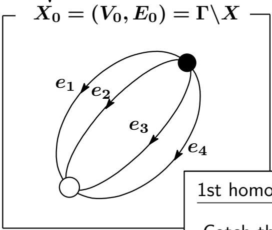
Realize $\boldsymbol{X}$ into $\boldsymbol{\Gamma} \otimes \mathbb{R}$ by using
(1) (modified) harmonic realization $\boldsymbol{\Phi_0}$
(2) Albanese metric $\boldsymbol{g_0}$ on $\boldsymbol{\Gamma} \otimes \mathbb{R}$



$\boldsymbol{\Phi_0}$



$\boldsymbol{\pi}$ (Devide by $\boldsymbol{\Gamma} = \langle \boldsymbol{\sigma_1}, \boldsymbol{\sigma_2} \rangle$)



$\boldsymbol{\rho_{\mathbb{R}}}$

loop in $\boldsymbol{X_0} \longmapsto \boldsymbol{a\sigma_1} + \boldsymbol{b\sigma_2}$
e.g.) asymptotic direction $\boldsymbol{\rho_{\mathbb{R}}(\gamma_p)}$

1st homology group $\mathbf{H_1}(\boldsymbol{X_0}, \mathbb{R})$

Catch the quantative data

e.g.) homological direction $\boldsymbol{\gamma_p}$

♣ **In this talk, we discuss this problem for non-symmetric RWs on nilpotent covering graphs from a viewpoint of discrete geometric analysis developed by T. Sunada (with M. Kotani).**

Scheme 1 : Replace the usual transition operator by

"transition-shift operator"

to "delete" the diverging drift term. (cf. `arXiv:1806.03804`)

Scheme 2 : Introduce a one-parameter family of RWs on X

$$(\Omega_x(X), \mathbb{P}_x^{(\varepsilon)}, \{w_n^{(\varepsilon)}\}_{n=0}^{\infty}) \qquad (0 \leq \varepsilon \leq 1)$$

to "weaken" the diverging drift term. ($\longrightarrow$ This talk !)

♠ **"Scheme 2" is applied to the study of the hydrodynamic limit of weakly asymmetric exclusion processes.**

Nilpotent Lie group (as a continuous model)

♠ How to realize the Γ-nilpotent covering graph X into some continuous space ?

[Malćev ('51)]

$\exists G = G_\Gamma$: connected & simply connected nilpotent Lie group such that Γ is isomorphic to a cocompact lattice in $(G, \cdot)$.

♠ By a certain deformation of the product $\cdot$ on G, we may assume that G is a stratified Lie group of step r.
Namely, its Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ satisfies

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}^{(i)}; \quad [\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \begin{cases} \subset \mathfrak{g}^{(i+j)} & (i+j \le r), \\ = \{0_{\mathfrak{g}}\} & (i+j > r), \end{cases}$$

and $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(i)}]$ $(i = 1, \dots, r-1)$.

♠ **Example:** 3-dim Heisenberg group $\mathbb{H}^3(\mathbb{R})(= \mathbb{G}^{(2)}(\mathbb{R}^2))$

▷ $\Gamma = \mathbb{H}^3(\mathbb{Z}) := \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{Z} \right\} \Big(\underset{\text{lattice}}{\hookrightarrow} G = \mathbb{H}^3(\mathbb{R}) \Big).$

▷ $\mathfrak{g} = \mathrm{Lie}(G) = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$

▷ $X_1 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, X_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, X_3 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

$$\Big(\rightsquigarrow [X_1, X_2] = X_3, \ [X_1, X_3] = [X_2, X_3] = 0_{\mathfrak{g}}. \Big)$$

▷ $G = \mathbb{H}^3(\mathbb{R})$: a (free) nilpotent Lie group of step 2, i.e.,

$$\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)}; \ \mathfrak{g}^{(1)} = \mathrm{span}_{\mathbb{R}}\{X_1, X_2\}, \ \mathfrak{g}^{(2)} := \mathrm{span}_{\mathbb{R}}\{X_3\}.$$

♠ We identify G with $\mathbb{R}^n$ through the canonical coordinates of the 1st kind:

$$G \ni \exp\Big(\sum_{k=1}^{r} \underbrace{\sum_{i=1}^{d_k} x_i^{(k)} X_i^{(k)}}_{\in \mathfrak{g}^{(k)}} \Big)$$

$$\longleftrightarrow (x^{(1)}, x^{(2)}, \dots, x^{(r)}) \in \mathbb{R}^{d_1+d_2+\cdots+d_r},$$

where

- ▷ $\mathfrak{g} = (\mathfrak{g}^{(1)}, g_0) \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$.
- ▷ $\mathfrak{g}^{(k)} = \mathrm{span}_{\mathbb{R}}\{X_1^{(k)}, \dots, X_{d_k}^{(k)}\}$ $(k = 1, \dots, r)$.
- ▷ $x^{(k)} = (x_1^{(k)}, \dots, x_{d_k}^{(k)}) \in \mathbb{R}^{d_k} \cong \mathfrak{g}^{(k)}$ $(k = 1, \dots, r)$.

Construction of the Albanese metric on $\mathfrak{g}^{(1)}$

♠ **We induce a special flat metric on $\mathfrak{g}^{(1)}$, called the Albanese metric, by the following diagram:**

$$
\begin{tikzcd}
(\mathfrak{g}^{(1)}, g_0) \arrow[d, leftrightarrow, "\text{dual}"] & \mathrm{H}_1(X_0,\mathbb{R}) \arrow[l, twoheadrightarrow, "\rho_{\mathbb{R}}"'] \arrow[d, leftrightarrow, "\text{dual}"] \\
\mathrm{Hom}(\mathfrak{g}^{(1)},\mathbb{R}) \arrow[r, hookrightarrow, "{}^t\rho_{\mathbb{R}}"'] & \mathrm{H}^1(X_0,\mathbb{R}) \cong \big(\mathcal{H}^1(X_0), \langle\!\langle \cdot,\cdot \rangle\!\rangle_p\big).
\end{tikzcd}
$$

▷ $\mathcal{H}^1(X_0) := \Big\{\omega \in C^1(X_0,\mathbb{R}) \,:\, \sum_{e \in (E_0)_x} p(e)\omega(e) = \langle \gamma_p, \omega\rangle\Big\}$

$$\text{with } \langle\!\langle \omega,\eta \rangle\!\rangle_p := \sum_{e\in E_0} \widetilde{m}(e)\omega(e)\eta(e) - \langle\gamma_p,\omega\rangle\langle\gamma_p,\eta\rangle.$$

▷ $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0,\mathbb{R}) \twoheadrightarrow \mathfrak{g}^{(1)}$ **: the surjective linear map defined by**

$$\rho_{\mathbb{R}}([c]) := \log(\sigma_c)\big|_{\mathfrak{g}^{(1)}} \quad \text{for } [c] \in \mathrm{H}_1(X_0,\mathbb{R})$$

$$\text{s.t. } \sigma_c \in \Gamma(\hookrightarrow G) \text{ satisfies } \sigma_c \cdot o(\tilde{c}) = t(\tilde{c}) \text{ on } X.$$

Harmonic realization of the graph X into G

- We consider a Γ-equivariant map $\Phi : X = (V, E) \longrightarrow G$:

$$\Phi(\gamma x) = \gamma \cdot \Phi(x) \quad (\gamma \in \Gamma,\, x \in V).$$

Definition [Modified Harmonic Realization]

A realization $\Phi_0 : X \longrightarrow G$ is said to be **modified harmonic** if

$$\Delta\Big(\log \Phi_0\big|_{\mathfrak{g}^{(1)}}\Big)(x) = \rho_{\mathbb{R}}(\gamma_p) \quad (x \in V),$$

where $\Delta := L - I$: the discrete Laplacian on X.

- Such Φ_0 is uniquely determined up to $\mathfrak{g}^{(1)}$-translation, however, it has the **ambiguity** in $(\mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)})$-component!
- $\rho_{\mathbb{R}}(\gamma_p)$ is called the **($\mathfrak{g}^{(1)}$-)asymptotic direction** of the RW.
  (LLN): $\lim_{n\to\infty} \frac{1}{n} \log \Phi_0(w_n) = \rho_{\mathbb{R}}(\gamma_p)$
- We emphasize that

$$\gamma_p = 0 \quad \overset{\Longrightarrow}{\not\Longleftarrow} \quad \rho_{\mathbb{R}}(\gamma_p) = 0_{\mathfrak{g}}.$$

Dilation & the CC-metric on G

♠ We introduce the 1-parameter group of dilations $\{\tau_\varepsilon\}_{\varepsilon \geq 0}$ on G:

$$G \ni (x^{(1)}, x^{(2)}, \dots, x^{(r)}) \overset{\tau_\varepsilon}{\longmapsto} (\varepsilon x^{(1)}, \varepsilon^2 x^{(2)}, \dots, \varepsilon^r x^{(r)}) \in G.$$

♠ We equip G with the Carnot-Carathéodory metric:

$$d_{\mathrm{CC}}(g,h) := \inf \Big\{ \int_0^1 \|\dot{c}(t)\|_{g_0}\, dt \,\Big|\, c \in \mathrm{AC}([0,1];G), \; c(0) = g,\ c(1) = h,\ \dot{c}(t) \in \mathfrak{g}^{(1)}_{c(t)} \Big\} \quad (g,h \in G).$$

♠ (G, d_{CC}) is not only a metric space but also a geodesic space.

Family of RWs with weak asymmetry

♠ For $0 \le \varepsilon \le 1$, we define

$$p_\varepsilon(e) := p_0(e) + \varepsilon q(e) \qquad (e \in E),$$

where

$$p_0(e) := \frac{1}{2}\Big(p(e) + \frac{m\big(t(e)\big)}{m\big(o(e)\big)} p(\overline{e})\Big) : m\text{-symmetric},$$

$$q(e) := \frac{1}{2}\Big(p(e) - \frac{m\big(t(e)\big)}{m\big(o(e)\big)} p(\overline{e})\Big) : m\text{-anti-symmetric}.$$

♠ Namely, p_ε is defined by the linear interpolation between the symmetric transition probability p_0 and the given one $p = p_1$.

▷ RW on X : $(\Omega_x(X), \mathbb{P}_x^{(\varepsilon)}, \{w_n^{(\varepsilon)}\}_{n=0}^{\infty}) \quad (0 \le \varepsilon \le 1)$.

♠ $m_\varepsilon(e) = m(e), \quad \gamma_{p_\varepsilon} = \varepsilon \gamma_p \quad (0 \le \varepsilon \le 1)$.

♠ For $0 \le \varepsilon \le 1$,

▷ $$L_{(\varepsilon)}f(x) = \sum_{e \in E_x} p_\varepsilon(e) f\big(t(e)\big) \quad (x \in V,\ f : V \longrightarrow \mathbb{R}).$$

▷ $g_0^{(\varepsilon)}$: Albanese metric on $\mathfrak{g}^{(1)}$ associated with p_ε.

($\Longrightarrow$ Continuity of the Albanese metric $g_0^{(\varepsilon)}$ w.r.t. ε.)

▷ $G_{(\varepsilon)}$: nilpotent Lie group whose Lie algebra is

$$(\mathfrak{g}^{(1)}, g_0^{(\varepsilon)}) \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)}.$$

▷ $\Phi_0^{(\varepsilon)} : X \longrightarrow G$: (p_ε-)modified harmonic realization, i.e.,

$$(L_{(\varepsilon)} - I)\Big(\log \Phi_0^{(\varepsilon)}\big|_{\mathfrak{g}^{(1)}}\Big)(x) = \varepsilon \rho_{\mathbb{R}}(\gamma_p) \quad (x \in V).$$

▷ $P_\varepsilon : C_\infty(G) \longrightarrow C_\infty(X)$: scaling operator defined by

$$P_\varepsilon f(x) := f\Big(\tau_\varepsilon\big(\Phi_0^{(\varepsilon)}(x)\big)\Big) \qquad (x \in V,\ 0 \le \varepsilon \le 1).$$

Semigroup-CLT

- We actually have, for $f \in C_0^\infty(G_{(0)})$,

$$\frac{1}{N\varepsilon^2}(I - L^N_{(\varepsilon)})P_\varepsilon f \sim P_\varepsilon \mathcal{A}_{(\varepsilon)} f$$

as $N \to \infty$, $\varepsilon \searrow 0$ and $N^2\varepsilon \searrow 0$ in some sense, where

$$\mathcal{A}_{(\varepsilon)} = \underbrace{-\frac{1}{2}\sum_{i=1}^{d_1} V_i^2}_{\text{sub-Laplacian on } G_{(0)}} \underbrace{-\rho_{\mathbb{R}}(\gamma_p)}_{\in \mathfrak{g}^{(1)}} \underbrace{-\beta^{(\varepsilon)}(\Phi_0^{(\varepsilon)})}_{\in \mathfrak{g}^{(2)}},$$

$$\beta^{(\varepsilon)}(\Phi_0^{(\varepsilon)}) := \sum_{e \in E_0} \widetilde{m}_\varepsilon(e) \log\left(\Phi_0^{(\varepsilon)}(o(e))^{-1} \cdot \Phi_0^{(\varepsilon)}(t(e))\right)\Big|_{\mathfrak{g}^{(2)}}.$$

- Usually, the diverging drift term appears in $\mathfrak{g}^{(1)}$-direction. However, it is weakened due to $\gamma_{p_\varepsilon} = \varepsilon\gamma_p$.

$\boxed{\text{Question}}$ **What is the behavior of $\beta^{(\varepsilon)}(\Phi_0^{(\varepsilon)})$ as $\varepsilon \searrow 0$?**

♠ **Unfortunately, it is NOT expected that**

$$\lim_{\varepsilon \searrow 0} \Phi_0^{(\varepsilon)}(x) = \Phi_0^{(0)}(x) \qquad (x \in V).$$

▷ $\exists \{\Phi_0^{(\varepsilon)}\}_{0 \le \varepsilon \le 1}$ **s.t.** $\left\| \log \left(\Phi_0^{(\varepsilon)}(x)^{-1} \cdot \Phi_0^{(0)}(x) \right) \Big|_{\mathfrak{g}^{(k)}} \right\|_{\mathfrak{g}^{(k)}} \longrightarrow \infty$ **for** $k = 2, 3, \ldots, r$.

♠ **We now impose**

(A1)

$$\sum_{x \in \mathcal{F}} m(x) \Big\{ \log \Phi_0^{(\varepsilon)}(x) \Big|_{\mathfrak{g}^{(1)}} - \log \Phi_0^{(0)}(x) \Big|_{\mathfrak{g}^{(1)}} \Big\} = 0,$$

where $\mathcal{F}$: a fundamental domain of X.

Key Proposition

Under **(A1)**, we have $\lim_{\varepsilon \searrow 0} \beta^{(\varepsilon)}(\Phi_0^{(\varepsilon)}) = 0_{\mathfrak{g}}$.

♠ **By combining Proposition 1 with the Trotter approximation theorem, we obtain a semigroup-CLT.**

Theorem 1. (Ishiwata-K-Namba, '18)

Under **(A1)**, we have, for $0 \leq s \leq t$ and $f \in C_\infty(G)$,

$$\lim_{n \to \infty} \left\| L_{(n^{-1/2})}^{[nt]-[ns]} P_{n^{-1/2}} f - P_{n^{-1/2}} e^{-(t-s)\mathcal{A}} f \right\|_\infty = 0,$$

where $\mathcal{A}$ is a 2nd order sub-elliptic operator on $G_{(0)}$ defined by

$$\mathcal{A} = -\frac{1}{2} \sum_{i=1}^{d_1} V_i^2 - \rho_{\mathbb{R}}(\gamma_p).$$

▷ $\{V_1, V_2, \ldots, V_{d_1}\}$: ONB of $(\mathfrak{g}^{(1)}, g_0^{(0)})$.

Functional CLT

▷ $x_* \in V$: a reference point s.t. $\Phi_0^{(0)}(x_*) = 1_G$.
(**Note** : It is not always $\Phi_0^{(\varepsilon)}(x_*) = 1_G$ due to **(A1)**.)

♠ We define

$$\mathcal{X}_t^{(\varepsilon,n)}(c) := \tau_\varepsilon\Big(\Phi_0^{(\varepsilon)}(w_{[nt]}^{(\varepsilon)})(c)\Big)$$

for $0 \le t \le 1$, $0 \le \varepsilon \le 1$, $n = 1, 2, \dots$ and $c \in \Omega_{x_*}(X)$.

♠ We also define a $G_{(0)}$-valued continuous stochastic process $\mathcal{Y}^{(\varepsilon,n)} = (\mathcal{Y}_t^{(\varepsilon,n)})_{0\le t\le 1}$ by the d_{CC}-geodesic interpolation (w.r.t. $g_0^{(0)}$-metric) of $\{\mathcal{X}_{k/n}^{(\varepsilon,n)}\}_{k=0}^n$ for every $0 \le \varepsilon \le 1$.

♣ To show tightness of $\{\mathcal{Y}^{(n^{-1/2},n)}\}$, we need to impose an additional assumption on $(\Phi_0^{(\varepsilon)})_{0\le\varepsilon\le 1}$.

(A2)

$\exists\, C > 0$ s.t. for $k = 2, 3, \ldots, r$,

$$\sup_{0 \le \varepsilon \le 1} \max_{x \in \mathcal{F}} \left\| \log\left(\Phi_0^{(\varepsilon)}(x)^{-1} \cdot \Phi_0^{(0)}(x) \right)\Big|_{\mathfrak{g}^{(k)}} \right\|_{\mathfrak{g}^{(k)}} \le C.$$

- Intuitively, the situations that the "distance" between $\Phi_0^{(\varepsilon)}(x)$ and $\Phi_0^{(0)}(x)$ tends to be too big as $\varepsilon \searrow 0$ are removed under **(A2)**.
- Under **(A1)** & **(A2)**, we can show

$$\mathbb{E}^{\mathbb{P}_{x_*}^{(n^{-1/2})}}\left[d_{\mathrm{CC}}\left(\mathcal{Y}_t^{(n^{-1/2},n)}, \mathcal{Y}_s^{(n^{-1/2},n)} \right)^{4m} \right] \le C(t-s)^{2m}$$

by combining the modified harmonicity of $\Phi_0^{(n^{-1/2})}$, several martingale inequalities and an idea (of the proof) of Lyons' extension theorem in rough path theory.

Theorem 2. (Ishiwata-K-Namba, '18)

Under **(A1)** & **(A2)**, we obtain, for all $\alpha < 1/2$,

$$(\mathcal{Y}_t^{(n^{-1/2},n)})_{0\le t\le 1} \underset{n\to\infty}{\Longrightarrow} (Y_t)_{0\le t\le 1} \quad \text{in } C^{0,\alpha}\big([0,1],G_{(0)}\big).$$

▷ $(Y_t)_{0\le t\le 1}$: $G_{(0)}$-valued diffusion process which solves

$$dY_t = \sum_{i=1}^{d_1} V_i(Y_t) \circ dB_t^i + \rho_{\mathbb{R}}(\gamma_p)(Y_t)\,dt, \qquad Y_0 = 1_G.$$

▷ $C^{0,\alpha}\big([0,1],G_{(0)}\big) := \overline{\mathrm{Lip}\big([0,1];G_{(0)}\big)}^{\|\cdot\|_{\alpha\text{-Höl}}}$: Polish, where

$$\|w\|_{\alpha\text{-Höl}} := \sup_{0\le s<t\le 1} \frac{d_{\mathrm{CC}}(w_s,w_t)}{|t-s|^{\alpha}} + d_{\mathrm{CC}}(1_G,w_0).$$

Some Comments

♠ $\exists$ many results on CLTs in which

$$\text{sub-Laplacian} + \mathfrak{g}^{(2)}\text{-valued drift}$$

is captured as the generator of the limiting diffusion.

- Raugi ('78), Pap ('93), Alexopoulos ('02), ...

♠ Indeed, we obtained such a (functional) CLT for non-symmetric RWs on X by applying Scheme 1 (transition-shift scheme). As the generator of the limiting diffusion, we have

$$\mathcal{A} = -\frac{1}{2}\sum_{i=1}^{d_1} V_i^2 - \beta_\rho(\Phi_0), \quad \text{where}$$

$$\beta_\rho(\Phi_0) := \sum_{e \in E_0} \widetilde{m}(e) \log\Big(\Phi_0\big(o(e)\big)^{-1} \cdot \Phi_0\big(t(e)\big) \cdot e^{-\rho_{\mathbb{R}}(\gamma_p)}\Big)\Big|_{\mathfrak{g}^{(2)}}.$$

♠ Note that $\gamma_p = 0 \Longrightarrow \beta_\rho(\Phi_0) = 0_{\mathfrak{g}}$.
(This drift arises from the non-symmetry of the given RW.)

♠ However, to our best knowledge, there seems to be few results on CLTs in the nilpotent setting in which a $\mathfrak{g}^{(1)}$-valued drift appears in the generator of the limiting diffusion.

♡ (Namba, in preparation): By combining Schemes 1 & 2, i.e.,

$$\begin{aligned} \mathcal{L}_{(\varepsilon)} f(x,t) &= \sum_{e \in E_x} p_\varepsilon(e) f\big(t(e), t+1\big), \\ \mathcal{P}_\varepsilon f(x,t) &= f\Big(\tau_\varepsilon\Big(\Phi_0^{(\varepsilon)}(x) \cdot \exp(tb)\Big)\Big), \quad b \in \mathfrak{g}^{(2)}, \end{aligned}$$

we also obtain a CLT with $\mathcal{A} = -\frac{1}{2}\sum_{i=1}^{d_1} V_i^2 - \rho_{\mathbb{R}}(\gamma_p) - b$.

♡ Applying the ($\mathfrak{g}^{(1)}$-)corrector-method (which is standard in stochastic homogenization), we might prove Theorem 2 without the modified-harmonicity of $\Phi_0^{(\varepsilon)}$.