Central limit theorem for random walks on nilpotent covering graphs with weak asymmetry

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Long time behaviors of random walks (RWs) on an infinite graph is a well-studied topic in geometry, harmonic analysis and graph theory, to say nothing of probability theory. It is known that geometric features such as the *periodicity* and the *volume growth* of the underlying graph affect long time behaviors of RWs. By putting an emphasis on them, Ishiwata, Kawabi and Kotani [1] considered a non-symmetric random walk $\{w_n\}_{n=0}^{\infty}$ on a Γ-crystal lattice X, a covering graph of a finite graph whose covering transformation group Γ is abelian. Through a discrete analogue of the harmonic map from X into a Euclidean space $\Gamma \otimes \mathbb{R}$, they established two kinds of functional central limit theorems (CLTs) for $\{w_n\}_{n=0}^{\infty}$. In fact, since a diverging drift term arising from the non-symmetry prevents us from taking the CLT-scaling limit directly, it is difficult to prove such CLTs. To overcome the difficulty, two schemes were introduced in [1]. One is to replace the usual transition operator by the transition-shift operator to "delete" the diverging drift term. The other is to introduce a family of non-symmetric RWs on X to "weaken" the diverging drift term. (The latter scheme is also applied in the study of the hydrodynamic limit of *weakly asymmetric* simple exclusion processes.)

Let Γ be a finitely generated nilpotent group. In [2], we considered a non-symmetric RW $\{w_n\}_{n=0}^{\infty}$ on a Γ-*nilpotent covering graph*, a generalization of both crystal lattices and Cayley graphs of a finitely generated group of polynomial volume growth. By extending the former scheme to the nilpotent case, we established a functional CLT for $\{w_n\}_{n=0}^{\infty}$ in [2]. The main purpose of this talk is to extend the latter scheme to the nilpotent case and to establish another functional CLT for $\{w_n\}_{n=0}^{\infty}$. This talk is based on our recent preprint [3].

Let $X = (V, E)$ be a Γ-nilpotent covering graph. Here V is the set of all vertices and E the set of all oriented edges in X. For $e \in E$, we denote the origin, terminus and inverse edge of e by $o(e), t(e)$ and $\overline{e}$, respectively. We set $E_x := \{e \in E \,|\, o(e) = x\}$ for $x \in V$. Let $p : E \longrightarrow (0, 1]$ be a Γ-invariant transition probability and $(\Omega_x(X), \mathbb{P}_x, \{w_n\}_{n=0}^{\infty})$ a RW on X starting from $x \in V$ associated with p. Through the covering map $\pi : X \longrightarrow X_0$, we also consider the RW $(\Omega_{\pi(x)}(X_0), \mathbb{P}_{\pi(x)}, \{\pi(w_n)\}_{n=0}^{\infty})$ and the corresponding transition probability is also denoted by $p : E_0 \longrightarrow (0, 1]$. We denote by $m : V_0 \longrightarrow (0, 1]$ the normalized invariant measure on X_0 and also write $m : V \longrightarrow (0, 1]$ for the Γ-invariant lift of m to X. Let $\mathrm{H}_1(X_0, \mathbb{R})$ be the first homology group of X_0. We define the *homological direction* of the RW on X_0 by $\gamma_p := \sum_{e \in E_0} p(e)m\big(o(e)\big)e \in \mathrm{H}_1(X_0, \mathbb{R})$. We call the RW on X_0 (m-)*symmetric* if $p(e)m\big(o(e)\big) = p(\overline{e})m\big(t(e)\big)$ for $e \in E_0$. Otherwise, it is called (m-)*non-symmetric*. Note that the RW on X_0 is (m-)symmetric if and only if $\gamma_p = 0$.

Thanks to the celebrated theorem of Malćev, we find a connected and simply connected nilpotent Lie group G such that Γ is isomorphic to a cocompact lattice in G. The nilpotent Lie group G is equipped with the canonical dilations $(\tau_\varepsilon)_{\varepsilon \geq 0}$, which gives a scalar multiplication on

G. By realizing X into G, CLTs for RWs on X can be discussed. Let $\mathfrak{g}$ be the corresponding Lie algebra of G and $\mathfrak{g}^{(1)} \cong G/[G,G]$ the generating part of $\mathfrak{g}$. We take a canonical surjective linear map $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{R}) \longrightarrow \mathfrak{g}^{(1)}$ by using the general theory of covering spaces. Thanks to the map $\rho_{\mathbb{R}}$ and the discrete Hodge–Kodaira theorem, a flat metric g_0 associated with the transition probability p, called the *Albanese metric*, is induced on $\mathfrak{g}^{(1)}$. A periodic realization $\Phi_0 : X \longrightarrow G$ is said to be *modified harmonic* if

$$\sum_{e \in E_x} p(e) \log\Big(\Phi_0\big(o(e)\big)^{-1} \cdot \Phi_0\big(t(e)\big)\Big)\Big|_{\mathfrak{g}^{(1)}} = \rho_{\mathbb{R}}(\gamma_p) \qquad (x \in V).$$

The quantity $\rho_{\mathbb{R}}(\gamma_p) \in \mathfrak{g}^{(1)}$ is called the *asymptotic direction*, which also appears in the law of large numbers for $\mathfrak{g}^{(1)}$-valued RW $\{\log\big(\Phi_0(w_n)\big)\big|_{\mathfrak{g}^{(1)}}\}_{n=0}^{\infty}$. It should be noted that $\gamma_p = 0$ implies $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}_{\mathfrak{g}}$, however, the converse does not hold in general.

For the given transition probability p, we introduce a family of Γ-invariant transition probabilities $(p_\varepsilon)_{0 \le \varepsilon \le 1}$ on X by $p_\varepsilon(e) := p_0(e) + \varepsilon q(e)$ for $e \in E$, where

$$p_0(e) := \frac{1}{2}\Big(p(e) + \frac{m\big(t(e)\big)}{m\big(o(e)\big)} p(\overline{e})\Big), \quad q(e) := \frac{1}{2}\Big(p(e) - \frac{m\big(t(e)\big)}{m\big(o(e)\big)} p(\overline{e})\Big).$$

Namely, the family $(p_\varepsilon)_{0 \le \varepsilon \le 1}$ is given by the linear interpolation between the given transition probability $p = p_1$ and the (m-)symmetric probability p_0. Moreover, the homological direction γ_{p_ε} equals $\varepsilon\gamma_p$ for every $0 \le \varepsilon \le 1$, which plays a key role in the proof of main theorems.

We now fix a reference point $x_* \in V$ such that $\Phi_0^{(0)}(x_*) = \mathbf{1}_G$, where $\mathbf{1}_G$ is the unit element of G. We write $g_0^{(\varepsilon)}$ for the Albanese metric on $\mathfrak{g}^{(1)}$ associated with p_ε and $\Phi_0^{(\varepsilon)} : X \longrightarrow G$ be the (p_ε-)modified harmonic realization for every $0 \le \varepsilon \le 1$. We set $\mathcal{Y}_{k/n}^{(\varepsilon,n)}(c) := \tau_{n^{-1/2}}\big(\Phi_0^{(\varepsilon)}(w_k(c))\big)$ for $n \in \mathbb{N}$, $k = 0, 1, \ldots, n$, $c \in \Omega_{x_*}(X)$ and $0 \le \varepsilon \le 1$. We then define a G-valued continuous stochastic process $\mathcal{Y}^{(\varepsilon,n)} = (\mathcal{Y}_t^{(\varepsilon,n)})_{0 \le t \le 1}$ by the geodesic interpolation of $\{\mathcal{Y}_{k/n}^{(\varepsilon,n)}\}_{k=0}^n$ with respect to the Carnot–Carathéodory metric on G. We take an orthonormal basis $\{V_1, V_2, \ldots, V_{d_1}\}$ of $(\mathfrak{g}^{(1)}, g_0^{(0)})$ and consider a stochastic differential equation (SDE)

$$dY_t = \sum_{i=1}^{d_1} V_i(Y_t) \circ dB_t^i + \rho_{\mathbb{R}}(\gamma_p)(Y_t)\, dt, \qquad Y_0 = \mathbf{1}_G,$$

where $(B_t)_{0 \le t \le 1} = (B_t^1, B_t^2, \ldots, B_t^{d_1})_{0 \le t \le 1}$ is an $\mathbb{R}^{d_1}$-standard Brownian motion starting from $B_0 = \mathbf{0}$. Let $Y = (Y_t)_{0 \le t \le 1}$ be the G-valued diffusion process which solves the SDE above.

We now state our main result as follows:

Theorem *Under several natural assumptions on* $\{\Phi_0^{(\varepsilon)}\}_{0 \le \varepsilon \le 1}$*, the sequence* $\{\mathcal{Y}^{(n^{-1/2},n)}\}_{n=1}^{\infty}$ *converges in law to the diffusion process* Y *in* $C^{0,\alpha\text{-Höl}}([0,1]; G)$ *as* $n \to \infty$ *for all* $\alpha < 1/2$.

References

[1] S. Ishiwata, H. Kawabi and M. Kotani: J. Funct. Anal. **272** (2017), pp.1553–1624.

[2] S. Ishiwata, H. Kawabi and R. Namba: preprint (2018), `arXiv:1806.03804`.

[3] S. Ishiwata, H. Kawabi and R. Namba: preprint (2018), `arXiv:1808.08856`.