Nilpotent Groups, Heat Kernels and Closed Geodesics

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1. Heat kernels: Let M be a compact Riemannian manifolds or a finite unorieted graph. For a discrete group Γ, take a normal covering $\Pi : X \to M$ with the covering transformation group Γ. We denote by $k_X(t,p,q)$ the heat kernel (resp. the transition probability of simple random walks) of X when M is a Riemannian manifold (resp. a finite graph). We are interested in the following problem.

Problem 1. What is the asymptotic behavior of $k_X(t,p,q)$ as $t \to \infty$.

In the case when Γ is an infinite abelian group, the following results are known.

Theorem 2. *Let X be a normal covering of copact manifold with an abelian covering transformation group Γ.*

$$k_X(t,p,q) \sim \frac{1}{t^{r/2}}(c_0 + \frac{c_1}{t} + \frac{c_2}{t^2} + \ldots) \qquad \text{if} \quad t \to \infty,$$

where r is the rank of abelian group Γ and $c_0, c_1, c_2, \ldots$ are constants depending on the geometry of M.

Similar results hold for graphs with suitable modifications. This result is due to Kotani-Sunada. The leading terms of the above theorem is obtained by several authors. In particular, Kotani-Shirai-Sunada focus on the geometric nature of c_0 which is written in terms of the volume of "Jacobi torus". Our first result is the followings.

Theorem 3. *For the 3 dimensional discrete Heisenberg group Γ and Γ-covering $\Pi : X \to M$,*

$$k_X(t,p,q) \sim \frac{1}{t^2}(c_0 + \frac{c_1}{t} + \frac{c_2}{t^2} + \ldots) \qquad \text{if} \quad t \to \infty,$$

where $c_0, c_1, c_2 \ldots$ are constants depending on the geometry of M.

We conjecture that the similar results holds for general nilpotent groups Γ. In fact, in the case when M is a finite graph, the asymtotics for the leading term is obtained by a combination of Alexopoulos and Ishiwata for general nilpotent groups. However, their method seems not to give geometric nature of the c_0 but our methods would give them.

2. Closed geodesics: Let M be a compact Riemannian manifold of negative curvature and let $\pi(x)$ denote the number of prime closed geodesics on M whose length is at most x. Celebrated results of Selberg, Margulis, Parry-Pollicott asserts that

Theorem 4.

$$\pi(x) \sim \frac{e^{hx}}{hx},$$

where h is the topological entropy of the geodesics flow on M.

This is called prime geodesic theorem which is geometric analogue of prime number theorem. As a variants of this result, there are several results in geometry, which are analogue of the Dirichlet density theorem for arithmetic progressin or the Chebotarev density theorem for algebraic extension of number fields. Let us formulate the problem. For a finitely generated discrete group Γ, we consider a surjective homomorphism Φ from the fundamental group $\pi_1(M)$ of M to Γ. Taking into account that there is one to one corresponding between the set closed geodesics on M and the set of conjugacy classes of $\pi_1(M)$, for a conjugacy class α in Γ, let $\pi(x, \alpha)$ denote the number of prime closed geodesics γ on M whose length is at most x and satisfy $\Phi([\gamma]) \subset \alpha$, where $[\gamma]$ is corresponding conjugacy class of γ in $\pi_1(M)$. Then our problems is as follows;

Problem 5. What is the asymptotic behavior of $\pi(x, \alpha)$ as $x \to \infty$?

Note that when Γ is the trivial group, $\pi(x, \alpha) = \pi(x)$ and the answer of the above problem is given by prime geodesic theorem. First answer of the above problem is given when Γ is a finite group by Parry-Pollicott and Adachi-Sunada independently as follows;

Theorem 6.

$$\pi(x, \alpha) \sim \frac{\sharp\alpha}{\sharp\Gamma} \frac{e^{hx}}{hx},$$

where $\sharp\alpha$ and $\sharp\Gamma$ denote the cardinals of α and Γ respectively.

Next answer is given when Γ is an infinite Abelian group. In the case when M is a compact Riemann surface of genus g with constant negative curvature -1, the following asymptotic result is given by Phillips and Sarnak,

Theorem 7.

$$\pi(x, \alpha) \sim (g-1)^g \frac{e^x}{x^{g+1}} (1 + \frac{c_1}{x} + \frac{c_2}{+} \dots)$$

The case when Γ is a discrete nilpotent group, we have the following conjecture;

Conjecture 8. For a conjugacy class α of a central element of finitely generated discrete nilpotent group Γ,

$$\pi(x, \alpha) \sim C \frac{e^{hx}}{x^{r+1}},$$

where C is a constant depending on the geometry of M and the spectral properties of a hypoelliptic operator related to Γ and r is the polynomial growth order of Γ.

In some special case , we have the followings.

Theorem 9. *Let M be a compact Riemannian surface with the constant negative curvature -1 and Γ be the 3 dimensional discrete Heisenberg group. For a conjugacy class α of central element of Γ, we have the asymptotic expansion*

$$\pi(x, \alpha) \sim \frac{e^x}{x^3} (c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots),$$

where $c_0, c_1, c_2 \dots$ are constants depending on the geometry of M.

Note that in this case, the polynomial growth order $r = 4$ and the topological entropy $h = 1$. If a conjugacy class α is not coming from central elements, then the asymptotic behavior of $\pi(x, \alpha)$ can be reduced to the analysis of the case for abelian groups, which is

$$\pi(x, \alpha) \sim \frac{e^x}{x^2} (c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots),$$