INTEGRATION WITH RESPECT TO HÖLDER ROUGH PATHS OF ORDER GREATER THAN 1/4: AN APPROACH VIA FRACTIONAL CALCULUS

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This study is an alternative approach to the fundamental theory of rough path analysis on the basis of fractional calculus. In this talk, using fractional calculus, we will introduce an approach to the integral of paths controlled by β-Hölder rough paths with $\beta \in (1/4, 1/3]$. Our definition of the integral is given by the Lebesgue integrals for fractional derivatives (Eq. (1)). We will show that, for a geometric β-Hölder rough path and its controlled paths with some assumptions, our definition of the integral is consistent with the usual definition, given by the limit of the compensated Riemann–Stieltjes sums (Eq. (2)). We will also explain that this result provides an explicit expression of the rough integral of 1-forms against geometric β-Hölder rough paths.

Definition of the integral. Let $T > 0$, $\triangle := \{(s,t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$, and V be a normed space. Let $\lambda \in (0,1]$ and $\Psi \in C(\triangle, V)$ such that $\sup_{0 \leq s < t \leq T} \|\Psi_{s,t}\|_V (t-s)^{-\lambda} < \infty$. Take $\alpha \in (0, \lambda)$. We define $\mathcal{D}^{\alpha}_{s+}\Psi$ with $s \in [0,T)$ and $\mathcal{D}^{\alpha}_{t-}\Psi$ with $t \in (0,T]$ by $\mathcal{D}^{\alpha}_{s+}\Psi(s) := 0$,

$$\mathcal{D}^{\alpha}_{s+}\Psi(u) := \frac{1}{\Gamma(1-\alpha)}\left(\frac{\Psi_{s,u}}{(u-s)^{\alpha}} + \alpha \int_s^u \frac{\Psi_{v,u}}{(u-v)^{\alpha+1}}\,dv\right)$$

for $u \in (s,T]$ and $\mathcal{D}^{\alpha}_{t-}\Psi(t) := 0$,

$$\mathcal{D}^{\alpha}_{t-}\Psi(u) := \frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)}\left(\frac{\Psi_{u,t}}{(t-u)^{\alpha}} + \alpha \int_u^t \frac{\Psi_{u,v}}{(v-u)^{\alpha+1}}\,dv\right)$$

for $u \in [0,t)$, where Γ denotes the gamma function. Let $\beta \in (1/4, 1/3]$ and $X = (1, X^1, X^2, X^3)$ be a β-Hölder rough path in $\mathbb{R}^d$. Take $\alpha \in (0, \beta)$ and $t \in (0,T]$. We define functions $\mathcal{R}^{(1,\alpha)}_{t-}X$, $\mathcal{R}^{(2,\alpha)}_{t-}X$, and $\mathcal{R}^{(3,\alpha)}_{t-}X$ on $[0,t]$ inductively as follows: for $u \in [0,t]$,

$$\begin{aligned}
\mathcal{R}^{(1,\alpha)}_{t-}X(u) &:= \mathcal{D}^{\alpha}_{t-}X^1(u),\\
\mathcal{R}^{(2,\alpha)}_{t-}X(u) &:= \mathcal{D}^{2\alpha}_{t-}X^2(u) - \mathcal{D}^{\alpha}_{t-}(X^1 \otimes \mathcal{R}^{(1,\alpha)}_{t-}X)(u),\\
\mathcal{R}^{(3,\alpha)}_{t-}X(u) &:= \mathcal{D}^{3\alpha}_{t-}X^3(u) - \mathcal{D}^{2\alpha}_{t-}(X^2 \otimes \mathcal{R}^{(1,\alpha)}_{t-}X)(u) - \mathcal{D}^{\alpha}_{t-}(X^1 \otimes \mathcal{R}^{(2,\alpha)}_{t-}X)(u).
\end{aligned}$$

For $\psi \in C([0,T], V)$, we define $\delta\psi \in C(\triangle, V)$ as $\delta\psi_{s,t} := \psi_t - \psi_s$ for $(s,t) \in \triangle$. Let $Z = (Z^{(0)}, Z^{(1)}, Z^{(2)})$ be a path controlled by X with values in $\mathbb{R}^n$. Similarly, we define $\mathcal{R}^{(3,\alpha)}_{t-}Z$ by

$$\mathcal{R}^{(3,\alpha)}_{t-}Z(u) := \mathcal{D}^{3\alpha}_{t-}R^2_0(Z)(u) - \mathcal{D}^{2\alpha}_{t-}(R^1_1(Z)\mathcal{R}^{(1,\alpha)}_{t-}X)(u) - \mathcal{D}^{\alpha}_{t-}(\delta Z^{(2)}\mathcal{R}^{(2,\alpha)}_{t-}X)(u)$$

for $u \in [0,t]$. Here, $R^{2-l}_l(Z)_{s,t} := Z^{(l)}_t - \sum_{i=0}^{2-l} Z^{(l+i)}_s X^i_{s,t}$ for $l = 0,1$ and $(s,t) \in \triangle$. We set $\pi(a \otimes b \otimes c) := b \otimes a \otimes c$ for $a,b,c \in \mathbb{R}^d$ and $\mathcal{R}^{(3,\alpha)}_{t-}\hat{X}(u) := \mathcal{R}^{(3,\alpha)}_{t-}X(u) + \pi(\mathcal{R}^{(3,\alpha)}_{t-}X(u))$ for $u \in [0,t]$. Let $Y = (Y^{(0)}, Y^{(1)}, Y^{(2)})$ be a path controlled by X with values in $\mathbb{R}^{m \times n}$. We set

$$\Xi_{s,t} := Y^{(0)}_s(Z^{(0)}_t - Z^{(0)}_s) + Y^{(1)}_s Z^{(1)}_s X^2_{s,t} + Y^{(2)}_s Z^{(1)}_s X^3_{s,t} + Y^{(1)}_s Z^{(2)}_s (X^3_{s,t} + \pi(X^3_{s,t}))$$

for $(s,t) \in \triangle$. Take a real number γ such that $(1-\beta)/3 < \gamma < \beta$, and note that the inequalities $1-\gamma < 3\beta$, $1-2\gamma < 2\beta$, and $1-3\gamma < \beta$ hold. For $(s,t) \in \triangle$, we define $I_X(Y,Z)_{s,t} \in \mathbb{R}^m$ by

$$
\begin{aligned}
I_X(Y,Z)_{s,t} := \Xi_{s,t} &+ (-1)^{1-\gamma} \int_s^t D_{s+}^{1-\gamma}(\Phi^3_{\cdot,u})_{s+}(u) \mathcal{R}_{t-}^{(1,\gamma)} X(u)\, du \\
&+ (-1)^{1-2\gamma} \int_s^t D_{s+}^{1-2\gamma}(\Phi^2_{\cdot,u})_{s+}(u) \mathcal{R}_{t-}^{(2,\gamma)} X(u)\, du \\
&+ (-1)^{1-3\gamma} \int_s^t \mathcal{D}_{s+}^{1-3\gamma} \delta Y^{(0)}(u) \mathcal{R}_{t-}^{(3,\gamma)} Z(u)\, du \\
&+ (-1)^{1-3\gamma} \int_s^t \mathcal{D}_{s+}^{1-3\gamma} \delta(Y^{(2)}Z^{(1)})(u) \mathcal{R}_{t-}^{(3,\gamma)} X(u)\, du \\
&+ (-1)^{1-3\gamma} \int_s^t \mathcal{D}_{s+}^{1-3\gamma} \delta(Y^{(1)}Z^{(2)})(u) \mathcal{R}_{t-}^{(3,\gamma)} \hat{X}(u)\, du.
\end{aligned}
\tag{1}
$$

Here, $D_{s+}^{1-\gamma}(\Phi^3_{\cdot,u})_{s+}(u)$ and $D_{s+}^{1-2\gamma}(\Phi^2_{\cdot,u})_{s+}(u)$ are defined as follows:

$$
\Phi^3_{v,u} := (Y_v^{(0)} + Y_v^{(1)} X^1_{v,u} + Y_v^{(2)} X^2_{v,u}) Z_u^{(1)} - Y_v^{(2)} X^2_{v,u} \delta Z^{(1)}_{v,u} - Y_v^{(1)} X^1_{v,u} R^1_1(Z)_{v,u}
$$

for $(v,u) \in \triangle$,

$$
D_{s+}^{1-\gamma}(\Phi^3_{\cdot,u})_{s+}(u) := \frac{1}{\Gamma(1-(1-\gamma))} \left(\frac{\Phi^3_{u,u} - \Phi^3_{s,u}}{(u-s)^{1-\gamma}} + (1-\gamma) \int_s^u \frac{\Phi^3_{u,u} - \Phi^3_{v,u}}{(u-v)^{(1-\gamma)+1}}\, dv \right)
$$

for $u \in [s,T]$ and

$$
\begin{aligned}
\Phi^2_{v,u} := &(Y_v^{(0)} + Y_v^{(1)} X^1_{v,u}) Z_u^{(2)} + (Y_v^{(1)} + Y_v^{(2)} X^1_{v,u}) Z_u^{(1)} \\
&- Y_v^{(2)} \delta Z^{(1)}_{v,u} X^1_{v,u} - Y_v^{(1)} R^1_1(Z)_{v,u} - Y_v^{(1)} X^1_{v,u} \delta Z^{(2)}_{v,u}
\end{aligned}
$$

for $(v,u) \in \triangle$,

$$
D_{s+}^{1-2\gamma}(\Phi^2_{\cdot,u})_{s+}(u) := \frac{1}{\Gamma(1-(1-2\gamma))} \left(\frac{\Phi^2_{u,u} - \Phi^2_{s,u}}{(u-s)^{1-2\gamma}} + (1-2\gamma) \int_s^u \frac{\Phi^2_{u,u} - \Phi^2_{v,u}}{(u-v)^{(1-2\gamma)+1}}\, dv \right)
$$

for $u \in [s,T]$. We note that $\sup_{0\le v<u\le T} |\Phi^3_{u,u} - \Phi^3_{v,u}|(u-v)^{-3\beta}$ and $\sup_{0\le v<u\le T} |\Phi^2_{u,u} - \Phi^2_{v,u}|(u-v)^{-2\beta}$ are finite. Therefore, $D_{s+}^{1-\gamma}(\Phi^3_{\cdot,u})_{s+}(u)$ and $D_{s+}^{1-2\gamma}(\Phi^2_{\cdot,u})_{s+}(u)$ are well-defined from $1-\gamma < 3\beta$ and $1-2\gamma < 2\beta$, respectively. We also note that the equalities

$$
D_{s+}^{1-\gamma}(\Phi^3_{\cdot,u})_{s+}(u) = \mathcal{D}_{s+}^{1-\gamma}(R^2_0(Y) Z_u^{(1)} + Y^{(2)} X^2 \delta Z^{(1)} + Y^{(1)} X^1 R^1_1(Z))(u)
$$

and

$$
\begin{aligned}
D_{s+}^{1-2\gamma}(\Phi^2_{\cdot,u})_{s+}(u) = \mathcal{D}_{s+}^{1-2\gamma}(&R^1_0(Y) Z_u^{(2)} + R^1_1(Y) Z_u^{(1)} \\
&+ Y^{(2)} \delta Z^{(1)} X^1 + Y^{(1)} R^1_1(Z) + Y^{(1)} X^1 \delta Z^{(2)})(u)
\end{aligned}
$$

hold by definition for $u \in [s,T]$. Here, $R^1_0(Y)_{s,t} := Y_t^{(0)} - Y_s^{(0)} - Y_s^{(1)} X^1_{s,t}$ for $(s,t) \in \triangle$.

In this talk, under suitable assumptions on X and Z, we will show that for $(s,t) \in \triangle$,

$$
I_X(Y,Z)_{s,t} = \lim_{|\mathcal{P}|\to 0} \sum_{i=0}^{l-1} \Xi_{t_i,t_{i+1}},
\tag{2}
$$

where the limit is taken over all finite partitions $\mathcal{P} = \{t_0, t_1, \dots, t_l\}$ of the interval $[s,t]$ such that $s = t_0 < t_1 < \cdots < t_l = t$ and $|\mathcal{P}| := \max_{0\le i\le l-1} |t_{i+1} - t_i|$.