Paracontrolled quasi-geostrophic equation with space-time white noise

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Aim of the talk

We study the following stochastic QGE on $(0, \infty) \times \mathbf{T}^2$ for $\mathbf{R}$-valued $u = u(t, x)$:

$$\partial_t u = -(-\Delta)^{\theta/2} u + R^{\perp} u \cdot \nabla u + \xi \quad \text{with } u(0, \cdot) = u_0.$$

Here, $\theta \in (0, 2]$, $R^{\perp} = (R_2, -R_1)$ with $R_j := \partial_j (-\Delta)^{-1/2}$ being the jth Riesz transform on $\mathbf{T}^2$, and
$\xi = \xi(t, x)$ is $\mathbf{R}$-valued space-time WN on $\mathbf{R} \times \mathbf{T}^2$, i.e., a centered Gaussian random field with covariance

$$\mathbb{E}[\xi(t, x)\xi(s, y)] = \delta_0(t - s)\delta_0(x - y).$$

We discuss mild solutions only. For $F_t(\bullet) = F(t, \bullet)$, Write

$$I[F]_t := \int_0^t e^{-(t-s)(-\triangle)^{\theta/2}} F_s \, ds.$$

Note: I makes sense for any (bad) F.

So, QGE is actually understood in the mild sense:

$$u_t = e^{-t(-\triangle)^{\theta/2}} u_0 + I[R^\perp u \cdot \nabla u]_t + X_t,$$

where $X_t := I[\xi]_t$ is "OU-like" process.

But, this SPDE is ill-posed even in the best case i.e., $\theta = 2$.
The reason is as follows:

- If the nonlinear term $R^{\perp}u \cdot \nabla u$ were absent, the solution is X itself. Its regularity for a fixed t is $0^{-} := 0 - \delta \quad (\forall \delta > 0)$.
- A natural guess: Regularity of u_t cannot be better than that of X_t, 0^{-}

  $\Longrightarrow$ Regularity of $R_j u_t$ and $\partial_i u_t$ are 0^{-} and $(-1)^{-}$, resp.

  $\Longrightarrow$ Those are just distributions, not a functions.

  $\Longrightarrow$ The product $R^{\perp}u \cdot \nabla u = R_2 u_t \, \partial_1 u_t - R_1 u_t \, \partial_2 u_t$ cannot be defined.

$\exists$ 3 methods to deal with this kind of singular SPDE.

- Hairer's theory of regularity structures.
- Gubinelli-Imkeller-Perkowski's paracontrolled calculus, also known as theory of paracontrolled disributions.
- Kupiainen's theory based on renormalization group theory. (This one is a little bit different)

Both Hairer's and GIP's theory are descendants of Gubinelli's version of rough path theory (controlled path theory).

♠ The aim of the talk is to solve QGE locally in time by using GIP's paracontrolled calculus.
(Probably, the two other methods work, too.)

There is no “bible” for paracontrolled calculus.
It has been gradually developed.

[Deterministic Part] In formulating and solving (locally) the paracontrolled QGE, we follow Mourrat-Weber’s method (CMP ’17).

[Probabilistic Part] To lift the white noise ξ to a random “driver” of the paracontrolled QGE, we follow Gubinelli-Perkowski’s method (CMP ’17).

(Stochastic) Dissipative QGE

• Quasi-geostrophic equation describes atmospheric and oceanographic flows on 2-dim space.
• "dissipative" $\rightsquigarrow$ $(-\Delta)^{-\theta/2}$ -term
• Stochastic dissipative QGE already exists, but with multiplicative noise. (The multiplication operator makes the noise term better.)

[cf] Liu-Röckner-Zhu ('13),
Röckner-Zhu-Zhu ('15), Zhu-Zhu ('17), etc.

• Stochastic dissipative QGE with additive noise seems new.
• When $4/3 < \theta \leq 2$, this additve QGE is super-renormalizable (i.e., subcritical in Hairer's sense) and therefore $\exists$ a hope.

Our Main Result (in loose form)

We approximate ξ by a smeared noise ξ^ϵ with a parameter $0 < \epsilon < 1$ (by killing the high frequencies) so that $\xi^\epsilon \to \xi$ as $\epsilon \downarrow 0$ in an appropriate (Besov-type) topology.

Assume $7/4 < \theta \leq 2$. Consider the "approximating" equation:

$$\partial_t u^\epsilon = -(-\Delta)^{\theta/2} u^\epsilon + R^\perp u^\epsilon \cdot \nabla u^\epsilon + \xi^\epsilon$$

with $u(0, \cdot\,) = u_0$.
Then, $u^\epsilon \to \exists u$ (locally in time) as $\epsilon \downarrow 0$ in an appropriate topology. (No renormalization is needed for this model.)

Some comments are in order.

- ∃ Global solution?
- The case $4/3 < \theta < 7/4$ is still open. Could be solvable (to some extent), but very cumbersome.
- There are almost paper on singular SPDE with fractional Laplacian. The only exception seems to be Gubinelli-Imkeller-Perkowskii (Pi, 15) on "fractional Burgers Eq." with $5/3 < \theta \leq 2$.
- We paved the way for "singular SPDE with fractional Laplacian."
  Hope it turns out to be useful for other equations.

Three steps of proof

[0] Heuristic computations to find which "symbols" (made of $X = I[\xi]$ alone) are needed (e.g., $I[\nabla X]$, $I[R^{\perp}X \cdot \nabla X]$, etc).

[1] Deterministic (local) solution theory. For a given deterministic "vector of symbols" $\vec{X} = (X, I[\nabla X], I[R^{\perp}X \cdot \nabla X], \ldots)$, define an "integration map" w.r.t. $\vec{X}$ on RHS of mild QGE, using paraproduct theory.

[2] Probabilistic part. Lifting OU-like process X to a random "vector of symbols" $\vec{X} = (X, I[\nabla X], I[R^{\perp}X \cdot \nabla X], \ldots)$.

Symbols for QGE and their regularities

After a long heuristic computation, it turns out that only 7 symbols are necessary and sufficient for paracontrolled QGE:

$$X \quad \frac{1}{2}(\theta - 2), \qquad V := I[\nabla X] \quad \frac{3}{2}\theta - 2$$

$$Y := I[R^{\perp} X \cdot \nabla X], \quad 2\theta - 3,$$

$$R^{\perp} V \odot \nabla X, \quad \nabla V \odot R^{\perp} X \quad 2(\theta - 2)$$

$$R^{\perp} Y \odot \nabla X, \quad \nabla Y \cdot R^{\perp} X \quad \frac{5}{2}(\theta - 2)$$

- Denote $\vec{X} := (X, V, Y, \dots)$ (called a "driver").
- Regulairty α actually means α^{-}.

Set $\mathcal{C}^\alpha = \mathcal{B}^\alpha_{\infty\infty}(\mathbf{T}^2)$ (Besov sp.) and $C_T\mathcal{C}^\alpha = C([0,T], \mathcal{C}^\alpha)$. Precisely, $\vec{X} := (X, V, W, \ldots)$ just stands for a (deterministic) element of

$$C_T\mathcal{C}^{\frac{1}{2}(\theta-2)} \times (C_T\mathcal{C}^{\frac{3}{2}\theta-2})^2 \times C_T\mathcal{C}^{2\theta-3} \times \cdots$$

with very natural constraints between X and $V := I[\nabla X]$ (no other constraints).

- Such $\vec{X}$'s form a closed subset of the product Banach space (the space of drivers of paracontrolled QGL $=: \mathcal{X}_T$).

[Rem] A blue symbol is just a generic element of (the corresponding coordinate of) $\mathcal{X}_T$.

- For example, $I[R^\perp X \cdot \nabla X]$ just stands for the third coordinate of $\mathcal{X}_T$. The whole "$I[R^\perp X \cdot \nabla X]$" is one such symbol.
- Do not try to get a meaning of it (because there is none). In particular, it does NOT mean

"Time-convolution (with heat semigroup) applied to
the rotated Riesz transform of X times the gradient of X"

Here, X is the first coordinate of $\mathcal{X}_T$.
(However, we assumed such a relation between X and $V := I[\nabla X]$ only.)

Bony's Paradifferential calculus

- For $f \in \mathcal{C}^{\alpha}$, $g \in \mathcal{C}^{\beta}$, the product fg can be defined if and only if $\alpha + \beta > 0$. In that case, $fg \in \mathcal{C}^{\alpha \wedge \beta}$.

- Let $f, g : \mathbf{T}^2 \to \mathbf{R}$. Decompose fg as follows:

$$fg = f \otimes g + f \odot g + f \otimes g$$

where $f \otimes g := \sum_{i<j-1} \triangle_i f \triangle_j g$, $\quad f \odot g := \sum_{|i-j|\le 1} \triangle_i f \triangle_j g$. (Here, $\triangle_i$ the ith Littlewood-Paley block. It picks up the frequencies of size $\approx 2^j$.)

• Paraproduct $f \otimes g$ is always defined, but its regularity may not be so nice. Its behaviour is similar to g (if f is a function). Let $f \in \mathcal{C}^{\alpha}$, $g \in \mathcal{C}^{\beta}$. Then,
$f \otimes g \in \mathcal{C}^{\beta}$ (if $\alpha > 0$) and $f \otimes g \in \mathcal{C}^{\alpha+\beta}$ (if $\alpha < 0$).

• Resonant term $f \odot g$ can be defined iff fg can be defined ($\iff \alpha + \beta > 0$). But, its regularity is nice (if defined):
$f \odot g \in \mathcal{C}^{\alpha+\beta}$ (if $\alpha + \beta > 0$).

• Commutation relation: If $\alpha + \beta + \gamma > 0$ (and $\alpha < 1$),
$\mathrm{C}(\mathrm{f}, \mathrm{g}, \mathrm{h}) := (f \otimes g) \odot h - f(g \odot h) \in \mathcal{C}^{\alpha+\beta+\gamma}$ nice!

• Relation with Riesz transform: $R_j(f \otimes g) - f \otimes R_j g$ is nice.

Fractional heat semigroup

For $F \in C_T\mathcal{C}^\alpha$,

$$I[F]_t := \int_0^t e^{-(t-s)(-\triangle)^{\theta/2}} F_s \, ds.$$

Then, we have the following Schauder estimate:

- $I: C_T\mathcal{C}^\alpha \to C_T\mathcal{C}^{\alpha+\theta}$ for $\forall \alpha \in \mathbf{R}$.

(I applies to any bad F and increases its regularity by θ.)

- We also have: $I[F \otimes G] = F \otimes I[G] +$ "nice"

PC Ansatz (or How to find a PC-QGE)

For simplicity, set $\theta = 2$ and $u_0 \equiv 0$. Let $\vec{X}$ be given.
Change the unknown by $\Psi := u - X$. Then, QGE becomes

$$\begin{aligned} \Psi &= I[R^\perp(X+\Psi)\cdot\nabla(X+\Psi)] \qquad +1 \\ &= I[R^\perp X\cdot\nabla X] + I[\underbrace{R^\perp\Psi\cdot\nabla X}_{ill-def}] + I[\underbrace{R^\perp X\cdot\nabla\Psi}_{ill-def}] + \text{nice.} \end{aligned}$$

We set RHS $=: \Gamma[\Psi]$. Look for a solution of the form

$$\Psi = \underbrace{I[R^\perp X\cdot\nabla X]}_{+1} + \underbrace{A}_{+1} \otimes \underbrace{I[\nabla X]}_{+1} + \underbrace{Q}_{+2}$$

for <u>some A, Q.</u> ("paracontrolled distribution w.r.t. $\vec{X}$")

♣ Heuristic meaning of PC ansatz:

$$\Psi = \underbrace{I[R^{\perp}X \cdot \nabla X]}_{+1} + \overbrace{\underbrace{A}_{+1} \otimes \underbrace{I[\nabla X]}_{+1}}^{+1} + \underbrace{Q}_{+2}$$

- Leading term with $+1$ is automatically determined.
- Remainder term with $+2$ is arbitrary since regularity is nice enough.
- Some freedom for the middle term with $+1$. Only those which look similar to $I[\nabla X]$ are allowed.

If Ψ of this form is substituted in Γ, then $\Gamma(\Psi)$ is not only well-def., but it also becomes PC-dist w.r.t. $\vec{X}$. As a result,

$$\Psi = \Gamma(\Psi) \qquad \text{fix point problem}$$

makes sense in the Banach sp. of PC-dist w.r.t. $\vec{X}$.

Note: If a PC-distribution Ψ is a fixed point, then necessarily

$$A = R^{\perp}\Psi$$

(This comes from $I[R^{\perp}\Psi \otimes \nabla X] = R^{\perp}\Psi \otimes I[\nabla X] + nice$. No other terms of type $A \otimes I[\nabla X]$ appears in $\Gamma(\Psi)$.)

Why $\Gamma(\Psi)$ well-defined? (Sketch) Think of $I[R^\perp\Psi \odot \nabla X]$ for

$$\Psi = \underbrace{I[R^\perp X \cdot \nabla X]}_{=:Y,\ +1} + \underbrace{A}_{+1} \circledless \underbrace{I[\nabla X]}_{+1} + \underbrace{Q}_{+2}$$

- Q's regularity is nice $\Rightarrow$ $Q \odot \nabla X$ is well-def.
- Y consists of X alone. So, we were able to postulate $Y \odot \nabla X$ beforehand.
- Since we have $R^\perp I[\nabla X] \odot \nabla X = R^\perp V \odot \nabla X$ beforehand,

$$\underbrace{R^\perp(\overbrace{A}^{+1} \circledless \overbrace{I[\nabla X]}^{+1})}_{+1} \odot \underbrace{\nabla X}_{-1} \qquad \text{(ill-def.)}$$

also makes sense, as we will see in the next slide.

By the commutation formula, we have

$$R^{\perp}(A \oslash I[\nabla X]) \odot \nabla X$$

$$\approx \quad (A \oslash R^{\perp} I[\nabla X]) \odot \nabla X$$

$$= \quad \underbrace{\overbrace{A}^{+1} \cdot \overbrace{(R^{\perp} I[\nabla X] \odot \nabla X)}^{0}}_{0} + \underbrace{C(\overbrace{A}^{+1}, \overbrace{R^{\perp} I[\nabla X]}^{+1}, \overbrace{\nabla X}^{-1})}_{+1}$$

Precisely, we interpret LHS as RHS when we formulate paracontrolled QGE. (This is not a mathematical fact.)

Mourrat-Weber's formulation

This fixed point problem looks like a system of PDEs.
Are Banach spaces of PC-distributions really needed?

- Change the unknown again by

$$(v + w :=) \quad \Psi - I[R^{\perp}X \cdot \nabla X] = u - X - I[R^{\perp}X \cdot \nabla X]$$

and rewrite QGE for the new unknown. Simply put,

$$\Psi - I[R^{\perp}X \cdot \nabla X] = \underbrace{A \otimes I[\nabla X]}_{\approx: v} + \underbrace{Q}_{\approx: w}$$

Then, the system of PDEs for v and w should be

$$\partial_t v = \underbrace{R^\perp(I[R^\perp X \cdot \nabla X] + v + w)}_{=R^\perp \Psi} \otimes \nabla X,$$

$$\partial_t w = \text{“the REST”} \quad (\text{written in terms of } v, w \text{ and } \vec{X}).$$

So, for a given $\vec{X}$, the system makes sense without setting a $\vec{X}$-dependent Banach space of PC-dist. Recall also that

$$I[R^\perp \Psi \otimes \nabla X] = R^\perp \Psi \otimes I[\nabla X] + \text{“nice”}.$$

- This can be solved by standard fixed point argument.

- MW's formulation is nice in the sense that it is not so different from the usual mild solution theory in PDE (except that we have strange-looking input data such as $I[\nabla X], \quad I[R^{\perp} X \cdot \nabla X], \ldots$).

- Dependency on $\vec{X}$ and the initial value can easily be tracked. This way, we prove local well-posedness (deterministically).

- There are probably good chances for PDE people.

Probabilistic Part (Lift of White Noise)

In this section, stationary OU-like process is used.
So, we slightly modify the definition of I:

$$I[F]_t := \int_{-\infty}^{t} e^{-(t-s)(-\triangle)^{\theta/2}} F_s \, ds.$$

Set $X_t := I[\xi]_t$ (stationary OU-like process).
$X_t^\epsilon = I[\xi^\epsilon]_t$: mollified process (high frequencies are killed).

♠ The stationary OU is easier to lift than non-stationary ones.
(A small drawback is the initial condition needs adjusting.)

Since $X^\epsilon(t,\bullet)$ is C^∞, the random driver

$$\vec{X}^\epsilon := (X^\epsilon, I[\nabla X^\epsilon], I[R^\perp X^\epsilon \cdot \nabla X^\epsilon], \ldots)$$

can be defined in the literal way, e.g., $I[R^\perp X^\epsilon \cdot \nabla X^\epsilon]$ means

"Time-convolution (with heat semigroup) applied to
the rotated Riesz transform of X^ϵ times the gradient of X^ϵ"

♣ Fortunately, $\vec{X}^\epsilon$ converges in the space of drivers, due to nice cancelations. To get a glimpse of this, think of

$$R_2 e^{2\pi i n\cdot x} \cdot \partial_1 e^{2\pi i n\cdot x} - R_1 e^{2\pi i n\cdot x} \cdot \partial_2 e^{2\pi i n\cdot x} = \frac{n_2 n_1 - n_1 n_2}{|n|} = 0.$$

$\longrightarrow$ Renormalization is NOT needed.

Write $\vec{X}^{\infty} := \lim_{\epsilon \searrow 0} \vec{X}^{\epsilon}$
(L^p-sense as $\mathcal{X}_T$-valued r.v. for $\forall p \in [1, \infty)$).

Let u^{ϵ} be a unique time-local solution of QGE in the usual sense:

$$u^{\epsilon}_t = e^{-t(-\triangle)^{\theta/2}} u_0 + I[R^{\perp} u^{\epsilon} \cdot \nabla u^{\epsilon}]_t + X^{\infty}_t.$$

Then, by the well-posedness of PC-QGE,
u^{ϵ} converges locally in time to a unique local solution to the PC-QGE driven by $\vec{X}^{\infty}$:

$$u^{\infty}_t = e^{-t(-\triangle)^{\theta/2}} u_0 + I[R^{\perp} u^{\infty} \cdot \nabla u^{\infty}]_t + X^{\infty}_t.$$