Local estimates of iterated paraproducts

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Recent breakthroughs for singular SPDEs

Singular SPDEs with renormalizations:

- Kardar-Parisi-Zhang equation

$$(\partial_t - \Delta)h = (\partial_x h)^2 - \infty + \xi, \quad (t,x) \in (0,\infty) \times \mathbb{R},$$

- Φ^4_d stochastic quantization model ($d = 2, 3$)

$$(\partial_t - \Delta)\Phi = -\Phi^3 + \infty\Phi + \xi, \quad (t,x) \in (0,\infty) \times \mathbb{R}^3.$$

Recent breakthroughs:

- Regularity structures (RS) (Hairer, 2014).
- Paracontrolled calculus (PC) (Gubinelli-Imkeller-Perkowski, 2015).

RS vs. PC

Solving SDE $dX = f(X)dB$:

Local behaviour

$$X_t - X_s = f(X_s)(B_t - B_s) + O(|t-s|^{1-})$$

$\longrightarrow$

Universal properties

- Lattice approximation
- Strong Feller property

$\Vert$ Relation?

Paracontrolled distribution

$$X = f(X) \circledless B + (C^{1-})$$

$\longrightarrow$

Particular properties

- Global well-posedness
- Exponential ergodicity

Relation between RS and PC

Previous researches:

- Gubinelli-Imkeller-Perkowski (2015)

  $\mathcal{R}f$ (reconstruction) $=$ Pf (extended Bony's paraproduct) $+ (\mathcal{C}^{\gamma})$.
- Martin-Perkowski (2018) showed another formulation of modelled distributions by P.

$\leftarrow$ Still based on the local behaviours, so not simplify the theory of RS.

Theorem (Bailleul-H)

RP theory	*RS*		*PC*
Rough path	*Model* (Π, Γ)	$\xleftrightarrow{\text{true}}$	*Drivers* $\{[\tau]\}$
Controlled path	*Modelled dist.*	$\xrightarrow{\text{true}}$ / $\xleftarrow{\text{unsolved}}$	*Paracontrolled dist.*

Modelled distribution ⇒ Paracontrolled distribution

Proposition (Bailleul-H)

Let $\gamma > 0$. For every γ-class modelled distribution

$$f = \sum_{|\tau|<\gamma} f_\tau \tau,$$

we have

$$f_\sigma = \sum_{|\sigma|<|\tau|<\gamma} f_\tau \otimes [\tau/\sigma] + (C^{\gamma-|\sigma|})$$

for every $\sigma \in \mathcal{B}$ s.t. $|\sigma| < \gamma$.

- This is the first “algebraic” formulation of PC.
- “MD $\Leftarrow$ PD” is not easy, because PD is compressed rather than MD, cf.

$$f \in C^\alpha \quad \Leftrightarrow \quad \exists \{f_k\}_{|k|<\alpha}, \text{ s.t. } f_0 = f,\ F(x) = \sum_{|k|<\alpha} f_k(x) \frac{X^k}{k!} \in \mathcal{D}^\alpha.$$

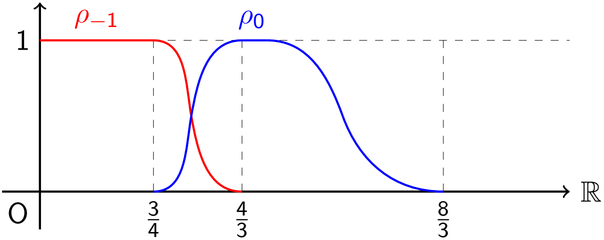
How to recover MD from PD?

Our approach is not general, but constructive.

1. From the PD from, each f_τ is the sum of the iterated paraprodusts
$$(\cdots((f_{\tau_n} \varolessthan [\tau_n/\tau_{n-1}]) \varolessthan [\tau_{n-1}/\tau_{n-2}]) \varolessthan \cdots \varolessthan [\tau_2/\tau_1]) \varolessthan [\tau_1/\tau].$$
We show the local estimates of such iterated paraproducts.
   - Extended Taylor series.
   - Multidimensional "rough integral".
2. We construct the particular Hopf algebra which represents such structures, i.e, Hopf algebra where "MD $\Leftrightarrow$ PD" holds.
   - Extension of the Hopf algebra associated with (weakly) geometric RPs
3. (Future problem) Can we decompose the general Hopf algebra into the sum of such algebras?

Littlewood-Paley theory

- $\sum_{j=-1}^{\infty} \rho_j = 1$: dyadic partition of unity, i.e. $\rho_j = \rho_0(2^{-j}\cdot)$, $j \geq 0$,



- $\triangle_j f := \mathcal{F}^{-1}(\rho_j(|\cdot|)\mathcal{F}f)$, $f \in \mathcal{S}'(\mathbb{R}^d)$.

Bony's paraproduct

$$f \otimes g := \sum_{i<j-1} \triangle_i f \triangle_j g.$$

Iterated paraproducts

For any sequence $\{f_j\}_{j\geq -1}$, we define $f_{j-} := \sum_{i<j-1} f_i$.

Definition

Fix $f^1, f^2, f^3, \cdots \in \mathcal{S}'(\mathbb{R}^d)$.

- $f^i_j := \triangle_j f^i$.
- $(f^1, f^2)_j := f^1_{j-} f^2_j$.
- $(f^1, f^2, f^3)_j := (f^1, f^2)_{j-} f^3_j$.
- $(f^1, f^2, \dots, f^{n-1}, f^n)_j := (f^1, f^2, \dots, f^{n-1})_{j-} f^n_j$.

We define

$$(f^1, f^2, \dots, f^{n-1}, f^n) := \sum_j (f^1, f^2, \dots, f^{n-1}, f^n)_j.$$

Note that

$$(f^1, f^2) = f^1 \olessthan f^2, \quad (f^1, f^2, f^3) \neq (f^1 \olessthan f^2) \olessthan f^3.$$

Regularity

Besov norm ($B^{\alpha}_{\infty,\infty}$)

$$\|f\|_{C^\alpha} := \sup_j \left(2^{j\alpha} \|\triangle_j f\|_{L^\infty(\mathbb{R}^d)} \right).$$

Proposition

Let $\alpha > 0$.

$$u \in C^\alpha, \ v \in C^\beta \quad \Rightarrow \quad u \circledcirc v \in C^\beta.$$

Taylor reminder

$$R_{yx} f := f(y) - \sum_{|k|<\alpha} \partial^k f(x) \frac{(y-x)^k}{k!} \ (= O(|y-x|^\alpha)).$$

We only get $R_{yx}(u \circledcirc v) = O(|y-x|^\beta)$!

Main results

Let $f^i \in C^{\alpha_i}$ and $\alpha_i > 0$ $(i = 1, 2, \ldots, n)$.

Theorem (Bailleul-H)

- *If* $\alpha_1 \notin \mathbb{N}$, *then*
$$|R_{yx} f| \lesssim \|f^1\|_{C^{\alpha_1}} |y-x|^{\alpha_1}.$$
- *Assume* $\alpha_1 + \alpha_2 \notin \mathbb{N}$. *There exists* $\partial_*^k(f^1, f^2) \in C(\mathbb{R}^d)$ *such that, if we define*
$$\Delta_{yx}(f^1, f^2) := (f^1, f^2)(y) - \sum_{|k|<\alpha_1+\alpha_2} \partial_*^k(f^1, f^2)(x) \frac{(y-x)^k}{k!} - \left(\sum_{|l|<\alpha_1} \partial^l f^1(x) \frac{(y-x)^l}{l!} \right) R_{yx} f^2,$$
then $|\Delta_{yx}(f^1, f^2)| \lesssim \|f^1\|_{C^{\alpha_1}} \|f^2\|_{C^{\alpha_2}} |y-x|^{\alpha_1+\alpha_2}$.

Theorem (Bailleul-H)

- *Assume* $\alpha_1+\alpha_2+\cdots+\alpha_n\notin\mathbb{N}$. *There exists* $\partial_*^k(f^1,f^2,\dots,f^j)\in C(\mathbb{R}^d)$ $(j\le n)$ *such that, if we define*

$$
\begin{aligned}
\Delta_{yx}(f^1,f^2,\dots,f^n) := (f^1,f^2,\dots,f^n)(y) - T_{yx}(f^1,f^2,\dots,f^n)\\
- T_{yx}(f^1,f^2,\dots,f^{n-1})R_{yx}f^n\\
- T_{yx}(f^1,f^2,\dots,f^{n-2})\Delta_{yx}(f^{n-1},f^n)\\
\cdots\\
- T_{yx}(f^1)\Delta_{yx}(f^2,\dots,f^n),
\end{aligned}
$$

where

$$
T_{yx}(f^1,f^2,\dots,f^j) = \sum_{|k|<\alpha_1+\alpha_2+\cdots+\alpha_j} \partial_*^k(f^1,f^2,\dots,f^j)(x)\frac{(y-x)^k}{k!}
$$

then

$$
|\Delta_{yx}(f^1,f^2,\dots,f^n)| \lesssim \|f^1\|_{C^{\alpha_1}}\|f^2\|_{C^{\alpha_2}}\cdots\|f^n\|_{C^{\alpha_n}}|y-x|^{\alpha_1+\alpha_2+\cdots+\alpha_n}.
$$

Relation with rough integrals

- $\alpha_1, \alpha_2 \in (0,1)$ and $\alpha_1 + \alpha_2 > 1$:

$$(f^1, f^2)(y) = (f^1, f^2)(x) + f^1(x)(f^2(y) - f^2(x)) + \langle \partial_*(f^1, f^2)(x), y - x \rangle + O(|y-x|^{\alpha_1+\alpha_2}).$$

  Compared with the Young integral ($d = 1$)

$$\int_x^y f^1(z)df^2(z) = f^1(x)(f^2(y) - f^2(x)) + O(|y-x|^{\alpha_1+\alpha_2}).$$

- $\alpha_1, \alpha_2, \alpha_3 \in (0,1)$, $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3 < 1 < \alpha_1 + \alpha_2 + \alpha_3 < 2$.

$$(f^1, f^2, f^3)(y) = (f^1, f^2, f^3)(x) + (f^1, f^2)(x)(f^3(y) - f^3(x)) + f^1(x)((f^2, f^3)(y) - (f^2, f^3)(x)) + \langle \partial_*(f^1, f^2, f^3)(x), y - x \rangle + O(|y-x|^{\alpha_1+\alpha_2+\alpha_3}).$$

  Compared with the rough integral ($d = 1$)

$$\int_x^y Y d\mathbf{B} = Y_x B^1_{yx} + Y'_x B^2_{yx} + O(|y-x|^{3\alpha}).$$

Algebraic structure behind singular PDEs

- Lyons' RP theory (1990s):

  path on $\mathbb{R}^d$ $\to$ path on the nilpotent Lie group (Geometric RP).
- Gubinelli's RP theory (2004): RP and controlled path. ("Vector bundle" on a manifold.)
- Gubinelli's branched RP theory (2010):

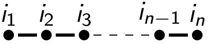
  Nongeometric RP $\leftrightarrow$ Connes-Kreimer Hopf algebra.

  Controlled path $\leftrightarrow$ Comodule.
- Hairer's RS theory (2014): calculus on a Hopf algebra H and a comodule over H.
- Bruned-Hairer-Zambotti algebra (2016): extended CK algebra representing "all" parabolic SPDEs.

  SPDE $\to$ "Black box" $\to$ Renormalized SPDE.

Hopf algebra behind rough path theory

- d-dim BM $\{B^i\}_{i=1}^d \leftrightarrow$ Alphabets $1, 2, \dots, d$.
- Geometric RP $\leftrightarrow$ Words.
  - $W = \{(i_1, \dots, i_n); n \geq 0, i_1, \dots, i_n \in \{1, \dots, d\}\}$.



$$\leftrightarrow \quad \int \left(\cdots \int \left(\int dB^{i_1} \right) dB^{i_2} \cdots \right) dB^{i_n}$$

  - Algebra $\mathcal{W} = \langle W \rangle$ with the shuffle product $\leftrightarrow$ Integration by parts.
  - $\Delta : \mathcal{W} \to \mathcal{W} \otimes \mathcal{W}$: cutting of words.
- Branched RP $\leftrightarrow$ Forests.
  - $T = \{$Rooted trees with indices $\in \{1, \dots, d\}$ at all vertices$\}$.

$$\leftrightarrow \quad \int \left(\int dB^j \right) \left(\int dB^k \right) dB^i$$

  - Commutative algebra $\mathcal{T}$ freely generated by T.
  - $\Delta : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$: admissible cuts.
- Our Hopf algebra is a kind of extension of the algebra of words.

Construction: Words

Fix $f^i \in C^{\alpha_i}$ and $\alpha_i > 0$ $(i = 1, 2, \dots, n)$.

- $\{1, \dots, n\}$: alphabets.
- $W = \{(i_1, \dots, i_k); k \geq 0, i_1, \dots, i_k \in \{1, \dots, n\}\}$, homogeneities

$$|(i_1, \dots, i_k)| := \alpha_{i_1} + \cdots + \alpha_{i_k}.$$

- $\mathcal{W}$: commutative algebra freely generated by W.
  (cf. $\mathcal{W}$/shuffle $\to$ Geometric RP.)
- Cutting map

$$\mathring{\Delta}(i_1, \dots, i_k) = \sum_{l=0}^{k} (i_{l+1}, \dots, i_k) \otimes (i_1, \dots, i_l).$$

Proposition

$(\mathcal{W}, \mathring{\Delta})$ *is a Hopf algebra.*

Construction: Derivatives and polynomials

Derivatives

- $\widetilde{W} = \{\tau_m := (\tau, m) \in W \times \mathbb{N}^d ; |\tau| - |m| > 0\}$, homogeneity
$$|\tau_m| := |\tau| - |m|.$$
- Extension
$$\mathring{\Delta}\tau_m := (\partial \otimes \mathrm{id} + \mathrm{id} \otimes \partial)^m \mathring{\Delta}\tau, \quad \partial^k \sigma := \sigma_k.$$

Polynomials on $\mathbb{R}^d$

- $P = \{X_1, \dots, X_d\}$, homogeneity $|X_i| = 1$.
- $\mathring{\Delta} X_i = X_i \otimes 1 + 1 \otimes X_i$.

Proposition

Let H be a commutative algebra freely generated by $\widetilde{W} \cup P$. Then $(H, \mathring{\Delta})$ is a Hopf algebra.

Construction: Twisting

- We define

$$\Delta := \exp(X \otimes \partial)\mathring{\Delta} := \sum_{m=0}^{\infty} \frac{1}{m!}\Big(\sum_i X_i \otimes \partial_i\Big)^m \mathring{\Delta},$$

  where

$$X_i : \tau \mapsto X_i\tau, \quad \partial_i : \begin{cases} \tau_m \mapsto \tau_{m+e_i}, \\ X^m \mapsto 0. \end{cases}$$

- Ex.

$$\Delta(1,2)_m = (1,2)_m \otimes 1 + \sum_k \frac{X^k}{k!} \otimes (1,2)_{m+k}$$
$$+ \sum \binom{m}{l} \frac{X^k}{k!}(2)_{m-l} \otimes (1)_{l+k}.$$

Theorem (Bailleul-H)

(H, Δ) is again a Hopf algebra.

Model on H

Recall $f^i \in C^{\alpha_i}$ and $\alpha_i > 0$ $(i = 1, 2, \ldots, n)$.

- Define the algebra homomorphism $g_x : H \to \mathbb{R}$ by

$$g_x((i_1, \ldots, i_k)_m) = \partial_*^m(f^{i_1}, \ldots, f^{i_k}), \quad g_x(X^m) = x^m.$$

- Define $\gamma_{yx} := (g_y \otimes g_x^{-1})\Delta$.

Theorem (Bailleul-H)

$\{\gamma_{yx}\}_{x,y\in\mathbb{R}^d}$ is a model on $\mathbb{R}^d$ for H, that is

$$\gamma_{yx}(\tau) = O(|y-x|^{|\tau|}), \quad \tau \in \widetilde{W} \cup P.$$

Modelled distribution on H

Theorem (Bailleul-H)

Let $g \in C^\beta$ with $\beta > 0$ and $\beta + \alpha_1 + \cdots + \alpha_n \notin \mathbb{N}$. Then the H-valued function

$$\mathbf{G}(x) = \sum_{|k|<\beta+\alpha_1+\cdots+\alpha_n} \partial_*^k(g, f^1, \dots, f^n)(x)\frac{X^k}{k!} + \sum_{|k|<\beta+\alpha_1+\cdots+\alpha_{n-1}} \partial_*^k(g, f^1, \dots, f^{n-1})(x)\frac{X^k}{k!}(n) + \cdots + \sum_{|k|<\beta} \partial^k g(x)\frac{X^k}{k!}(1, \dots, n)$$

is a $(\beta + \alpha_1 + \cdots + \alpha_n)$-modelled distribution on (H, γ).

MD $\Leftrightarrow$ PD on H

Theorem (Bailleul-H)

A modelled distribution

$$\mathbf{G}(x) = \sum_{|k|<\beta+\alpha_1+\cdots+\alpha_n} g^k(x)\frac{X^k}{k!} + \sum_{|k|<\beta+\alpha_1+\cdots+\alpha_{n-1}} g_n^k(x)\frac{X^k}{k!}(n)$$

$$+\cdots+\sum_{|k|<\beta} g_{1\ldots n}^k(x)\frac{X^k}{k!}(1,\ldots,n) \in \mathcal{D}^{\beta+\alpha_1+\cdots+\alpha_n}$$

is given by

$$\begin{cases} \quad g^0 = (h_1, f^1, \ldots, f^n) + (h_2, f^2, \ldots, f^n) + \cdots + (h_n, f^n) + h, \\ \quad g_n^0 = (h_1, f^1, \ldots, f^{n-1}) + (h_2, f^2, \ldots, f^{n-1}) + \cdots + h_n, \\ \quad \ldots \\ g_{2\ldots n}^0 = (h_1, f^1) + h_2, \\ g_{1\ldots n}^0 = h_1. \end{cases}$$

Future problems

- Can we replace

$$(f^1, f^2, \ldots, f^n) \quad \leftrightarrow \quad ((f^1 \otimes f^2) \otimes \cdots) \otimes f^n?$$

  (Our iterated paraproducts are fake.)

- Our algebra is an extension of the algebra of words. Can we similarly extend the CK algebra (algebra of forests)? Or can we represent such forest algebra by our algebra? (cf. Hairer-Kelly, 2015.)
- Can we decompose the general Hopf algebra into the sum of our algebras?