Identification from the SFCs of regularly Ogawa integrable random functions

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Notation and terminology

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$: filtrated probability space

$(B_t)_{t\in[0,1]}$: Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$

$(e_n)_{n\in\mathbb{N}}$: CONS of $L^2([0,1]\,;\mathbb{C})$

$f : [0,1]\times\Omega \to \mathbb{C}$ is random. $\underset{\mathrm{def}}{\Leftrightarrow}$ f is $\mathcal{L}([0,1])\otimes\mathcal{F}$-measurable.

For random functions X, Y and $t \in (0,1]$,

$\langle X, Y\rangle_t$: cross variation at t of X, Y (if exists),

$[X]_t = \langle X, X\rangle_t$: quadratic variation at t of X (if exists).

$$\operatorname{sgn} z = \begin{cases} 1 & , 0 \le \arg z < \pi \\ -1 & , -\pi \le \arg z < 0 \end{cases} \quad (\arg 0 := 0) \quad \text{for } z \in \mathbb{C}.$$

$\mathbb{K}$: $\mathbb{R}$ or $\mathbb{C}$

SFCs and Questions

SFC:

a: random function on $[0,1]$.

$(e_n, a\,dB)$: stochastic Fourier coefficient (SFC) of $a(t)$ defined by

$$(e_n, a\,dB) := \int_0^1 \overline{e_n(t)}a(t,\omega)\,dB_t.$$

Remark: Definition of SFC depends on how stochastic integral $\int dB_t$ is defined. But in this talk, assume $\int dB_t$ is Ogawa integral.

Question: Is $a(t)$ identified from SFCs $(e_n, a\,dB)$?

SFCs and Questions

Extension

SFC:

a, b : random functions on $[0,1]$.

dX: stochastic differential defined by $dX_t = a(t,\omega)\,dB_t + b(t,\omega)\,dt$.

(e_n, dX): stochastic Fourier coefficient (SFC) of dX defined by

$$
\begin{aligned}
(e_n, dX) &:= \int_0^1 \overline{e_n(t)}\,dX_t \\
&= \int_0^1 \overline{e_n(t)} a(t,\omega)\,dB_t + \int_0^1 \overline{e_n(t)} b(t,\omega)\,dt.
\end{aligned}
$$

Question: Are a, b identified from SFCs (e_n, dX) ?

Two notions of identification

- Identification with $(B_t)_{t\in[0,1]}$

  Identification by making use of values of the underlying Brownian motion $(B_t)_{t\in[0,1]}$.

- Identification without $(B_t)_{t\in[0,1]}$

  Identification by making no use of values of the underlying Brownian motion $(B_t)_{t\in[0,1]}$.

Previous works (Identification from SFC-Os)

- Identification with $(B_t)_{t\in[0,1]}$

  · Skorokhod integral processes (H., Kazumi[5])

  · Any random function of bounded variation (H.[1])

  · Certain $\mathbb{C}$-valued random functions (Ogawa,Uemura[4])

- Identification without $(B_t)_{t\in[0,1]}$

  · Non-negative AC random functions (Ogawa,Uemura[3])

  · Any non-negative random function of bounded variation (H.[1], extension of the result in [3])

  · Certain $\mathbb{C}$-valued random functions (Ogawa,Uemura[4]) (identification up to sign)

Present work : Identification from SFC-Os

- Identifiability : The family of random functions identified in some sense becomes a linear space.

$\Longrightarrow$

- Identification with $(B_t)_{t\in[0,1]}$

$$\text{Sums of } \begin{cases} \text{quasi-martingale} \\ \text{Skorokhod integral process} \\ \text{Hilbert-Schmidt integral transform of Wiener functional} \end{cases}$$

- Identification without $(B_t)_{t\in[0,1]}$

  The above random functions except their sign

・Extensions of the results in [1,5]

Definition of Ogawa integrals

Definition 1

$f \in L^0(\Omega\,; L^2[0,1])$

$\varphi = (\varphi_m)_{m\in\mathbb{N}}$: (ordered) CONS of $L^2([0,1]\,;\mathbb{K})$

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- Integral w.r.t. φ: $\int_0^1 f\, d_{\varphi} B$

$$\int_0^1 f(t)\, d_{\varphi} B_t := \sum_{m=1}^{\infty} \int_0^1 f(t)\varphi_m(t)\, dt \int_0^1 \varphi_m(t)\, dB_t \text{ in prob.}$$

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- Universal integral in $\mathbb{K}$: $\int_0^1 f \, d_{\mathrm{u}} B$

$$\int_0^1 f(t) \, d_{\mathrm{u}} B_t := \int_0^1 f(t) \, d_{\varphi} B_t \ \text{, if the RHS is independent of } \varphi.$$

Definition of Ogawa integrals

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Remark $\mathbb{K} = \mathbb{R}$ in this study.

Ogawa integrability

$$\mathcal{A} = \{\, f \in L^0([0,1] \times \Omega) \,|\, \mathrm{Re}\, f, \mathrm{Im}\, f \text{ are of bounded variation a.s.} \,\},$$

$$\mathcal{M} = \left\{ \int_0^{\cdot} g \, dB \,\middle|\, g \in L^0(\Omega\,; L^2[0,1]), \right.$$
$$\left. g \text{ is } (\mathcal{F}_t)_{t \in [0,1]}\text{-progressively measurable} \right\},$$

where $\int \, dB$: Itô integral.

Remark: f is quasi-martingale. $\underset{\text{def}}{\Leftrightarrow}$ $f \in \mathcal{A} + \mathcal{M}$.

Ogawa integrability

$\varphi = (\varphi_m)_{m \in \mathbb{N}}$: CONS of $L^2([0,1]\,;\mathbb{R})$

Regularity of CONS

φ is regular. $\underset{\text{def}}{\Leftrightarrow} \sup_{M \in \mathbb{N}} \left\| \sum_{m=1}^{M} \varphi_m \widetilde{\varphi}_m \right\|_{L^2[0,1]} < \infty \ \left(\widetilde{\varphi}_m(t) = \int_0^t \varphi_m(s)\,ds\right)$.

Theorem 1 (Ogawa[7](1985))

(1) Every quasi-martingale is φ-integrable. $\Leftrightarrow$ φ is regular.

(2) If (1) holds, the φ-integral of any quasi-martingale coincides with the symmetric integral (Ogawa, Storatonovich-Fisk).

Framework of Wiener chaos

$\mathcal{F}^B := \sigma(B_t \,|\, t \in [0,1])$

$\mathcal{L}_i^{r,2}$: Sobolev space in $L^2((\Omega, \mathcal{F}^B, P)\,; L^2[0,1]^i)$ with differentiability index r for each $i \in \{0\} \cup \mathbb{N}$

$D_t f(s)$: H-derivative of $f(s) \in \mathcal{L}_1^{1,2}$

$\int_0^1 f(t)\,\delta B_t$: Skorokhod integral of $f(t) \in \mathcal{L}_1^{1,2}$

$T_K f(t) := \int_0^1 K(t,s) f(s)\,ds$, $f \in \mathcal{L}_1^{1,2}$, $K \in L^2([0,1]^2)$

Ogawa integrability

Theorem 2 (H.,Kazumi[2,1](2018,2017))

Suppose

1. φ : regular CONS of $L^2([0,1]\,;\mathbb{R})$,
2. $e : [0,1] \to \mathbb{R}$; bounded variation,
3. $X_t = \int_0^t f(s)\,\delta B_s + T_K g(t) + X_0\ , t \in [0,1]$,
   where $f \in \mathcal{L}_1^{2,2}, g \in \mathcal{L}_1^{1,2}, X_0 \in \mathcal{L}_0^{1,2}, K \in L^2([0,1]^2)$.

Then, eX is φ-integrable and the integral is given by

$$\int_0^1 e(t)X_t\,\delta B_t + \frac{1}{2}\int_0^1 e(t)f(t)\,dt + \int_0^1 e(t)\Big(\int_0^t D_t\,\delta X_s + D_t X_0\Big)\,dt.$$

Previous results / Identification from SFC-$\mathrm{O_u}$'s

$a, b : [0,1] \times \Omega \to \mathbb{C}$; random

Remark: Always assume $b \in L^2[0,1]$ a.s.

Theorem 3 (H.[1](2017))

Suppose $a \in \mathcal{A}$.
Let $d_{\mathrm{u}} Y_t = a(t)\, d_{\mathrm{u}} B_t + b(t)\, dt$. Then,

(1) $|a|$ is identified without $(B_t)_{t\in[0,1]}$ from $((e_n, d_{\mathrm{u}} Y))_{n\in\mathbb{N}}$,

(2) a, b are identified with $(B_t)_{t\in[0,1]}$ from $((e_n, d_{\mathrm{u}} Y))_{n\in\mathbb{N}}$.

Remark 1 $|a|, a$ are reconstructed by Parseval-type transformation and law of iterated logarithm.
Remark 2 $|a|, a$ can be identified, even if $((e_n, d_{\mathrm{u}} Y))_{n\in\mathbb{N}}$ lacks its finite elements $(e_n, d_{\mathrm{u}} Y)$.
Remark 3 b is identified from $((e_n, d_{\mathrm{u}} Y))_{n\in\mathbb{N}}$ if $a(t)$ is identified.

Previous results / Identification from SFC-O$_\varphi$'s

$$\mathcal{W} = \left\{ \int_0^\cdot g \,\delta B \,\middle|\, g \in \mathcal{L}_1^{2,2} \right\} + \mathrm{span} \left\{ T_K h \,\middle|\, h \in \mathcal{L}_1^{1,2},\ \sup_{t\in[0,1]} |K(t,\cdot)|_{L^2[0,1]} < \infty \right\} \quad \subset \mathcal{L}_1^{1,2}$$

Theorem (H.,Kazumi[5,1](2018,2017))

Suppose

1. φ : regular CONS of $L^2([0,1]\,;\mathbb{R})$,
2. $\forall n \in \mathbb{N}\ e_n \in BV[0,1]$,
3. $a \in \mathcal{W}$,
4. $b \in \mathcal{L}_1^{0,2}$.

Then, a, b are identified from $((e_n\,,\,d_\varphi Y))_{n\in\mathbb{N}}$,
where $d_\varphi Y_t = a(t)\,d_\varphi B_t + b(t)\,dt$.

Previous results / Identification from SFC-O$_\psi$'s

Theorem (Ogawa, Uemura[8,4](2017))

Suppose $a(t)$ and CONSs $(e_n)_{n\in\mathbb{N}}$, $(\psi_m)_{m\in\mathbb{N}}$, $(\chi_k)_{k\in\mathbb{N}}$ satisfy the following:

1. $\exists(\lambda_k)_{k\in\mathbb{N}} : \forall k\, \lambda_k > 0,\ \sum_{k=1}^{\infty} \lambda_k < \infty,\ E\left(\sum_{k=1}^{\infty} \frac{1}{\lambda_k}\langle \chi_k, a\rangle^2_{L^2[0,1]}\right) < \infty.$
2. $\sup_{k\in\mathbb{N}, t\in[0,1]} |\chi_k(t)| < \infty.$
3. $\forall n, m \in \mathbb{N}\ e_n\psi_m \in L^2[0,1].$

Then,

(1) $(\operatorname{sgn} a)a$ is identified without $(B_t)_{t\in[0,1]}$ from $((e_n, a\, d_\psi B))_{n\in\mathbb{N}}$,

(2) $a(t)$ is identified with $(B_t)_{t\in[0,1]}$ from $((e_n, a\, d_\psi B))_{n\in\mathbb{N}}$.

Remark $(\operatorname{sgn} a)a$, a are reconstructed by Parseval-type transformation and cross variation.

Present work : Identification from SFC-O$_\varphi$'s

Main Assumption

1. CONS for Ogawa integral:

   φ : regular CONS of $L^2([0,1]\,;\mathbb{R})$

2. CONS for SFC:

   $\forall n \in \mathbb{N}\ \mathrm{Re}\, e_n, \mathrm{Im}\, e_n \in BV[0,1]$

$$\mathcal{W} = \left\{ \int_0^{\cdot} g\,\delta B \,\middle|\, g \in \mathcal{L}_1^{2,2} \right\} + \mathrm{span}\left\{ T_K h \,\middle|\, h \in \mathcal{L}_1^{1,2},\ \sup_{t\in[0,1]} |K(t,\cdot)|_{L^2[0,1]} < \infty \right\}$$

$\mathscr{L} = \mathcal{A} + \mathcal{M} + \mathcal{W} \cdots$ Family of φ-integrable random functions

Present work / Identifiability 1

Proposition 1

Suppose $a \in \mathscr{L} = \mathcal{A} + \mathcal{M} + \mathcal{W}$. Then, the following hold:

(1) $\underset{n\to\infty}{\text{l.i.p.}} \int_0^1 v_n a \, d_\varphi B = 0$ for every point sequence $(v_n)_{n\in\mathbb{N}}$ of $BV[0,1]$ which converges to 0 in L^2.

In particular,

(2) $$\left(\int_0^t a \, d_\varphi B = \sum_{n=1}^\infty \int_0^t e_n(s) \, ds \, (e_n, a \, d_\varphi B) \text{ in prob.} \right) \ \forall t \in [0,1].$$

Therefore,

(3) for a dense subset S of $[0,1]$ and $\mathscr{C} \subset \mathscr{L}$,

(i) $c \in \mathscr{C}$ is identified from $((e_n, c \, d_\varphi B))_{n\in\mathbb{N}}$.

$\Longleftrightarrow$

(ii) $c \in \mathscr{C}$ is identified from $(\int_0^t c \, d_\varphi B))_{t\in S}$.

Present work / Identifiability 1

Point of proof of (1):

- Case of $a \in \mathcal{A}$:

  Follow the proof of Theorem 3(H.[1](2017))
  (Itô-Nisio theorem, Doob's L^2-inequality).

- Case of $a = \int_0^\cdot g\, dB \in \mathcal{M}$:

  Step 1. Apply Itô formula and Theorem 1(Ogawa[7](1985)) for $v_n a$.
  Step 2. Apply limit properties of Itô integral and Lebesgue integral.

- Case of $a = \int_0^\cdot g\, \delta B + T_K h \in \mathcal{W}$:

  Step 1. Apply Theorem 2(H.,Kazumi[2,1](2018,2017)) for $v_n a$.
  Step 2. Apply Wiener chaos theory.

Identification by cross and quadratic variations

$$
\begin{aligned}
\mathscr{L}_{\mathrm{PC}}^{e,\varphi} = \Big\{\, a \in L^0([0,1]\times\Omega) \;\Big|\; & \Big(\int_0^t a\, d_\varphi B = \sum_{n=1}^{\infty} \int_0^t e_n(s)\, ds\, (e_n,\, a\, d_\varphi B) \text{ in prob. },\\
& \int_0^t |a(u)|^2\, du = \Big[\int_0^{\cdot} a\, d_\varphi B \Big]_t,\\
& \int_0^{t\wedge s} a(u)\, du = \Big\langle \int_0^{\cdot} a\, d_\varphi B,\, B_{\cdot\wedge s} \Big\rangle_t \Big) \; \forall s,t \in [0,1] \Big\}
\end{aligned}
$$

Identification by cross and quadratic variations

$$\mathscr{L}_{\mathrm{PC}}^{e,\varphi} = \Big\{ a \in L^0([0,1]\times\Omega) \,\Big|\, \Big(\int_0^t a\, d_\varphi B = \sum_{n=1}^{\infty} \int_0^t e_n(s)\, ds\, (e_n, a\, d_\varphi B) \text{ in prob. },\ \int_0^t |a(u)|^2\, du = \Big[\int_0^{\cdot} a\, d_\varphi B \Big]_t,\ \int_0^{t\wedge s} a(u)\, du = \Big\langle \int_0^{\cdot} a\, d_\varphi B,\, B_{\cdot\wedge s} \Big\rangle_t \Big)\ \forall s,t \in [0,1] \Big\}$$

Remark: For $a \in \mathscr{L}_{\mathrm{PC}}^{e,\varphi}$,

$$|a(t)| = \Big(\frac{d}{dt} \Big[\int_0^{\cdot} a\, d_\varphi B \Big]_t \Big)^{\frac{1}{2}}$$

$$a(t) = \frac{d}{dt} \Big\langle \int_0^{\cdot} a\, d_\varphi B,\, B \Big\rangle_t$$

Formulas on quadratic variation

Theorem 4 (D.Nualart, E.Pardoux[6](1988))

For $f \in \mathcal{L}_1^{1,2}$,

$$\left[\int_0^{\cdot} f \, \delta B \right]_t = \int_0^t |f(s)|^2 \, ds, \ \forall t \in [0,1].$$

Proposition 2

For $f \in \mathcal{A}$,

$$\left[\int_0^{\cdot} f \, \delta B \right]_t = \int_0^t |f(s)|^2 \, ds, \ \forall t \in [0,1].$$

Main results

Proposition 3

$\mathcal{A}, \mathcal{M}, \mathcal{W} \subset \mathscr{L}_{\mathrm{PC}}^{e,\varphi}$.

Theorem 5 (Identifiability 2)

$\mathscr{L}_{\mathrm{PC}}^{e,\varphi}$ becomes a vector space.

Corollary 1

Suppose $\mathrm{Re}\, a, \mathrm{Im}\, a \in \mathscr{L}_{\mathrm{PC}}^{e,\varphi}$.

Let $d_\varphi Y_t = a(t)\, d_\varphi B_t + b(t)\, dt$. Then,

(1) a, b are identified with $(B_t)_{t\in[0,1]}$ from $((e_n, d_\varphi Y))_{n\in\mathbb{N}}$,

(2) $|\mathrm{Re}\ a|$, $|\mathrm{Im}\ a|$, $\mathrm{Re}\ a\, \mathrm{Im}\ a$, $(\mathrm{sgn}\, a)a$ are identified without $(B_t)_{t\in[0,1]}$ from $((e_n, d_\varphi Y))_{n\in\mathbb{N}}$ by using $\forall f, g \in \mathscr{L}_{\mathrm{PC}}^{e,\varphi}\ \int_0^t f\overline{g}\, d\lambda = \left\langle \int_0^\cdot f\, d_\varphi B, \int_0^\cdot g\, d_\varphi B \right\rangle_t$.

Closure property w.r.t. sum of random functions with AC cross variation processes

Key to proof of Theorem 5:

Theorem 6

$$Q_{\mathrm{c}} = \left\{ X : [0,1] \to L^0(\Omega) \;\middle|\; \exists \hat{X} \in L^0(\Omega; L^2[0,1]) \right.$$
$$\left. \forall s,t \in [0,1]\ [X]_t = \int_0^t |\hat{X}|^2\, d\lambda,\ \langle X, B_{\cdot\wedge s}\rangle_t = \int_0^{t\wedge s} \hat{X}\, d\lambda \right\}$$

is a vector subspace of $L^0(\Omega)^{[0,1]}$, where λ is Lebesgue measure.

Sketch of proof:

Let $X, Y \in Q_{\mathrm{c}}$. Prove $[X+Y]_t = \int_0^t |\hat{X}+\hat{Y}|^2\, d\lambda$.

First, for $Z \in \mathcal{D} = \left\{ \sum_{j=1}^n r_j B_{\cdot\wedge t_j} 1_{A_j} \;\middle|\; r_j \in \mathbb{C}, t_j \in [0,1], A_j \in \mathcal{F} \right\} \subset Q_{\mathrm{c}}$,

$[\alpha X + \beta Z]_t = \int_0^t |\alpha\hat{X} + \beta\hat{Z}|^2\, d\lambda$, $\hat{Z} = \sum_{j=1}^n r_j 1_{[0,t_j]} \otimes 1_{A_j}$.

Closure property w.r.t. sum of random functions with AC cross variation processes

Next, approximate Y by $Y' \in \mathcal{D}$. $\Delta X := X_t - X_s$, $\Delta = (s,t)$.

Using the trivial identity:

$$(\Delta X + \Delta Y')^2 - (\Delta X + \Delta Y)^2 = (\Delta Y' - \Delta Y)^2 + 2(\Delta Y' - \Delta Y)(\Delta X + \Delta Y)$$

and Schwarz inequality, we have

$$\left| \sum (\Delta X + \Delta Y')^2 - \sum (\Delta X + \Delta Y)^2 \right|$$
$$\leq \sum (\Delta Y' - \Delta Y)^2 + 2\left(\sum (\Delta Y' - \Delta Y)^2 \right)^{\frac{1}{2}} \left(\sum (\Delta X + \Delta Y)^2 \right)^{\frac{1}{2}}.$$

Here $[X + Y']_t \underset{\hat{Y}' \to \hat{Y}}{\longrightarrow} \int_0^t |\hat{X} + \hat{Y}|^2 \, d\lambda$, $[Y' - Y]_t \underset{\hat{Y}' \to \hat{Y}}{\longrightarrow} 0$.

Therefore $[X + Y]_t = \int_0^t |\hat{X} + \hat{Y}|^2 \, d\lambda$. $\square$

References

[1] K. Hoshino, Identification of finite variation processes from the SFC. MSJ Autumn Meeting, abstract (2017)

[2] K. Hoshino, T. Kazumi, On the integrability of Ogawa integrals of noncausal Wiener functionals, to appear, Stochastics (2018)

[3] S. Ogawa, H. Uemura, On the identification of noncausal functions from the SFCs. RIMS Kôkyûroku. **1952**, 128-134 (2015)

[4] S. Ogawa, H. Uemura, On the reconstruction of random function from its SFCs defined by an arbitrary CONS. Symposium on Probability Theory, abstract (2017)

References

[5] K. Hoshino, T. Kazumi, On the identification of noncausal Wiener functionals from the stochastic Fourier coefficients, to appear, Journal of Theoretical Probability (2018)

[6] D. Nualart, E. Pardoux, Stochastic calculus with anticipating integrands. Probab. Th. Rel. Fields , 78, pp.535-581 (1988)

[7] S. Ogawa, The stochastic integral of noncausal type as an extension of the symmetric integrals. Japan J. Appl. Math. **2**, 229-240 (1985)

[8] S. Ogawa, H. Uemura, On the reconstruction of random function from its SFCs defined by an arbitrary CONS. MSJ Autumn Meeting, abstract (2017)