Discrete approximations for non-colliding SDEs

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Abstract

In this talk, we consider discrete approximations for non-colliding particle systems. We introduce a semi-implicit Euler-Maruyama approximation which preservers the non-colliding property for some class of non-colliding particle systems and provide a strong rate of convergence in L^p-norm. We also consider some modified explicit/implicit Euler-Maruyama schemes.

Non-colliding particle systems

A non-colliding particle systems $X = (X(t) = (X_1(t), \ldots, X_d(t)))_{t \geq 0}$ is a solution of the following system of stochastic differential equations (SDEs)

$$\mathrm{d}X_i(t) = \left\{ \sum_{j \neq i} \frac{\gamma_{i,j}}{X_i(t) - X_j(t)} + b_i(X_i(s)) \right\} dt + \sum_{j=1}^{d} \sigma_{i,j}(X(t)) \mathrm{d}W_j(t), \quad i = 1, \ldots, d, \tag{1}$$

with $X(0) \in \Delta_d = \{\mathbf{x} = (x_1, \ldots, x_d)^* \in \mathbb{R}^d : x_1 < x_2 < \cdots < x_d\}$, $\gamma_{i,j} = \gamma_{j,i} \geq 0$ and $W = (W(t) = (W_1(t), \ldots, W_d(t))^*)_{t \geq 0}$ a d-dimensional standard Brownian motion.

The existence and uniqueness of a strong non-colliding solution to (1) have been studied intensively by many others. However, there are still few results on the numerical approximation for such kind of systems. To the best of our knowledge, the paper of Li and Menon [4] is the only work in this direction. Li and Menon introduced an explicit "tamed" Euler-Maruyama approximation since the coefficient b_i are super linear growth. However, their scheme unfortunately does not preserve the non-colliding property of a solution, which is an important characteristic of the SDE (1).

Recently, many authors study numerical approximation for one-dimensional SDEs with boundary (e.g. Bessel process $\mathrm{d}X_t = dt/X_t + \mathrm{d}W_t, X_t > 0$ and CIR process $\mathrm{d}X_t = (a - bX_t)dt + X_t^{1/2}\mathrm{d}W_t, X_t > 0$). Dereich, Neuenkirch and Szpruch [2] introduced an implicit Euler-Maruyama scheme for CIR process and showed that the rate of convergence is $1/2$, and extended to one-dimensional SDEs with boundary condition by Alfonsi [1] and Neuenkirch and Szpruch [5].

Discrete approximations for non-colliding particle systems

Inspired by [1, 2, 5], we define a semi-implicit Euler-Maruyama scheme for a solution of non-colliding SDE (1) as follows: $X^{(n)}(0) := X(0)$ and for each $k = 0, \ldots, n-1$, $X^{(n)}(t_{k+1}^{(n)})$ is defined as the unique solution in Δ_d of the following equation:

$$X_i^{(n)}(t_{k+1}^{(n)}) = X_i^{(n)}(t_k^{(n)}) + \left\{ \sum_{j \neq i} \frac{\gamma_{i,j}}{X_i^{(n)}(t_{k+1}^{(n)}) - X_j^{(n)}(t_{k+1}^{(n)})} + b_i\left(X_i^{(n)}(t_k^{(n)}) \right) \right\} \frac{T}{n}$$
$$+ \sum_{j=1}^{d} \sigma_{i,j}\left(X^{(n)}(t_k^{(n)}) \right) \left\{ W_j(t_{k+1}^{(n)}) - W_j(t_k^{(n)}) \right\},$$

where $t_k^{(n)} := kT/n$. Since the equation

$$\xi_i = a_i + \sum_{j \neq i} \frac{c_{i,j}}{\xi_i - \xi_j}, \quad i = 1, \ldots, d,$$

has a unique solution in Δ_d for each $a_i \in \mathbb{R}$ and $c_{i,j} \geq 0$ with $c_{i,i+1} > 0$ (see Proposition 2.2 in [6]), thus $X^{(n)} = (X^{(n)}(t_k^{(n)}))_{k=0,\ldots,n}$ is well-defined for each $n \in \mathbb{N}$.

In this talk, under some assumptions on the constants $\gamma_{i,j}$ and the coefficients b_i, $\sigma_{i,j}$, we will show that the SDE (1) has a unique global strong solution on Δ_d and the Euler-Maruyama approximation $X^{(n)}$ converges to the unique solution to the non-colliding SDE (1) in L^p-sense for some $p \geq 1$ or 2 with convergence rate $n^{1/2}$ or n. More preciously, we will show that there exists $C > 0$ such that,

$$\mathbb{E}\left[\sup_{k=1,\ldots,n} |X(t_k^{(n)}) - X^{(n)}(t_k^{(n)})|^p \right]^{1/p} \leq \begin{cases} Cn^{-1/2}, & \text{if } b_i \text{ are Lipschitz continuos and } p \geq 1, \\ Cn^{-1}, & \text{if } b_i \in C_b^2(\mathbb{R};\mathbb{R}) \text{ and } p \geq 2, \end{cases}$$

(see Theorem 2.8 and 2.9 in [6]). Note that the singular coefficients $\frac{1}{x_i - x_j}$ make the system difficult to deal with. In order to overcome this obstacle, we need an upper bound for both moments and inverse moments of $X_i(t) - X_j(t)$.

References

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