Kolmogorov-Pearson diffusions and hypergeometric functions

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1 Introduction

We consider diffusuions generated by $\mathfrak{A} = a\frac{d^2}{dx^2} + b\frac{d}{dx}$. Here a is a quadratic function and b is a linear function. We call these diffusions as Kolmogorof-Pearson diffusions. We are interested in spectra of these generators. We want to determin all spectra completely. To do this, hypergeometric functions play a important role.

2 Sevral expressions of generators

Our generators are of the form

$$\mathfrak{A} = a\frac{d^2}{dx^2} + b\frac{d}{dx} \tag{1}$$

where a is quadtatic and b is linear. Following Feller, we can associate a measure dm and a function s. dm is called a speed measure and s is called a scale function. In our case, dm has a density ρ of the form $\rho = \exp\{\int (f/g)dx\}$ where f is linear and g is quadratic. We call this type of density as Pearson density. Pearson considered probability densities but we may admit infinite measure cases. s defines a measure ds and it has of the form $ds = \dfrac{1}{a\rho}dx$. Using a and ρ, b can be expressed as $b = a' + a(\log\rho)'$.

Now we can give several expressions of the generator as follows:

	generator	duality	differential opetaor
Kolmogorov	$a\dfrac{d^2}{dx^2} + b\dfrac{d}{dx}$		
Feller	$\dfrac{d}{dm}\dfrac{d}{ds}$	$\dfrac{d}{dm} = -\dfrac{d}{ds}^*$	$\dfrac{d}{ds}: L^2(dm) \to L^2(ds)$
Stein	$\left(a\dfrac{d}{dx} + b\right)\dfrac{d}{dx}$	$a\dfrac{d}{dx} + b = -\dfrac{d}{dx}^*$	$\dfrac{d}{dx}: L^2(\rho dx) \to L^2(a\rho dx)$

Using this, we can make following correspondences.

Feller's pair	$\dfrac{d}{dm}\dfrac{d}{ds} \longleftrightarrow \dfrac{d}{dm}\dfrac{d}{ds}$
Stein's pair	$(a\dfrac{d}{dx} + b)\dfrac{d}{dx} \longleftrightarrow \dfrac{d}{dx}(a\dfrac{d}{dx} + b)$

One important thing is that the class of Kolmogorov-Pearson diffusions are closed under Feller's pair and Stein's pair. From these pairings, we can show that

- If f is an eigenfunction, then so are f', $\frac{d}{ds}f$.
- If θ is an eigenfunction, then so are $a\theta' + b\theta$, $\frac{d}{dm}\theta$.

According to the degree of a, our generators are classified as

	complete family	incomplete family		special function
α-family	$a = 1$			F_1^0
β-family	$a = x$	$a = x^2$		F_1^1
γ-family	$a = x(1-x)$	$a = x(1+x)$	$a = 1 + x^2$	F_1^2

Further, associated speed measures are given as follows:

	complete family	incomplete family	
α-family	$e^{\beta x^2/2}$		
β-family	$x^\alpha e^{\beta x}$	$x^\alpha e^{\beta/x}$	
γ-family	$x^\alpha(1-x)^\beta$	$x^\alpha(1+x)^\beta$	$(1+x^2)^\alpha \exp\{2\beta \arctan x\}$

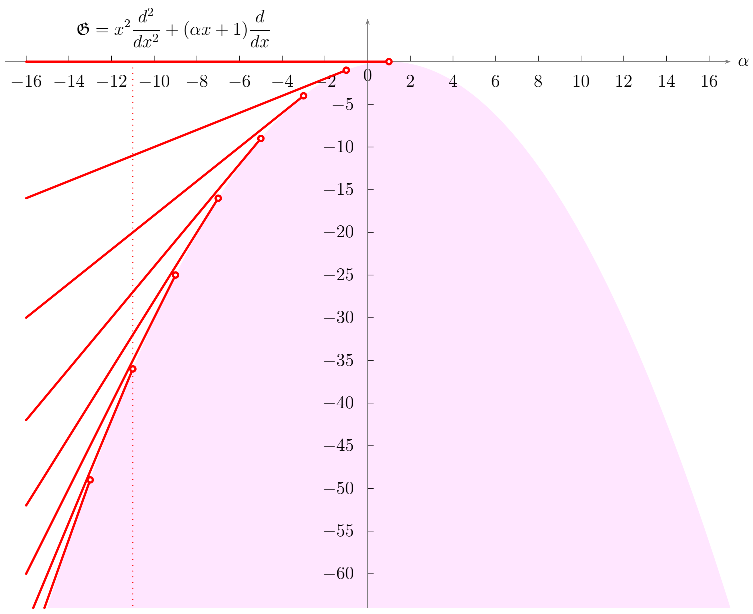
3 Spectra of generators

We have the following six cases:
(i) $a = 1$, (ii) $a = x$, (iii) $a = x^2$, (iv) $a = x(1-x)$, (v) $a = x(1+x)$, (vi) $a = 1 + x^2$.

We have discussed (i) and (ii) in the previous occasion. We will discuss here (iii) – (vi). In the case of (iii), the generator has the following form:

$$\mathfrak{A} = x^2 \frac{d}{dx^2} + (\alpha x - \beta)\frac{d}{dx}. \tag{2}$$

In particular, in the case $\beta = -1$, spectra are given as



Other cases will be discussed in the talk.