Multiray generalization of the arcsine laws for occupation times of infinite ergodic transformations

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1 Introduction

In this talk, we consider a certain distributional convergence of occupation time ratios for ergodic transformations preserving an infinite measure. We give a general limit theorem which can be regarded as a multiray extension of the 2-ray result by Thaler–Zweimüler [4]. Our general limit theorem can be applied to the following problems:

1. (Lamperti's process [2]) Let $Z = (Z_k)_{k\geq 0}$ be an irreducible and null-recurrent discretetime Markov chain on a countable discrete state space $\{0\} + \sum_{i=1}^{d} A_i$, where A_1, \ldots, A_d will be called the *rays*, having the following property: Z cannot skip the origin 0 when it changes rays, i.e., the condition $Z_n \in A_i$ and $Z_m \in A_j$ for some n < m and $i \neq j$ implies the existence of n < k < m for which $Z_k = 0$. Then, as $n \to \infty$,

$$\frac{1}{n} \left(\sum_{k=0}^{n-1} \mathbb{1}_{A_1}(Z_k), \dots, \sum_{k=0}^{n-1} \mathbb{1}_{A_d}(Z_k) \right) \to ?$$

- 2. (interval map with indifferent fixed points [3]) Let [0,1] be decomposed into $[0,1] = \sum_{i=1}^{d} I_i$ for disjoint intervals I_1, \ldots, I_d , and suppose that the map $T : [0,1] \to [0,1]$ satisfies the following conditions: for each i,
 - (a) $T|_{I_i}$ belongs to $C^2(I_i)$ and has a C^2 -extension over $\overline{I_i}$, and $\overline{T(I_i)} = [0, 1]$,
 - (b) there exists $x_i \in I_i$ such that

$$Tx_i = x_i, \quad T'x_i = 1, \quad \text{and} \\ (x - x_i)T''x > 0 \text{ for any } x \in I_i \setminus \{x_i\}.$$

In particular, T' > 1 on $I_i \setminus \{x_i\}$.

Let A_i 's be disjoint small neighborhoods of x_i 's, respectively, and take $Y := [0, 1] \setminus \sum_{i=1}^{d} A_i$. We will call A_1, \ldots, A_d the rays and Y the origin set. Note that we can take the rays sufficiently small so that the orbit $(T^k x)_{k\geq 0}$ cannot skip the origin set when it changes rays. In this setting, we know that $n^{-1} \sum_{k=0}^{n-1} \mathbb{1}_{\sum_{i=1}^{d} A_i}(T^k x) \to 1$, a.e., as $n \to \infty$. Then, as $n \to \infty$,

$$\frac{1}{n} \left(\sum_{k=0}^{n-1} \mathbb{1}_{A_1}(T^k x), \dots, \sum_{k=0}^{n-1} \mathbb{1}_{A_d}(T^k x) \right) \to ?$$

2 Main results

Let (X, \mathcal{A}, μ) be a standard measurable space with a σ -finite measure such that $\mu(X) = \infty$, and let $T : (X, \mathcal{A}, \mu) \to (X, \mathcal{A}, \mu)$ be a conservative, ergodic, measure preserving transformation (which is abbreviated by *CEMPT*). Assume that X is decomposed into

 $X = Y + \sum_{i=1}^{d} A_i$ for $Y \in \mathcal{A}$ with $\mu(Y) \in (0, \infty)$ and $A_i \in \mathcal{A}$ with $\mu(A_i) = \infty$ such that the orbit $(T^k x)_{k\geq 0}$ cannot skip the origin set Y when it changes rays A_1, \ldots, A_d . Set

$$S_n := \left(\sum_{k=0}^{n-1} \mathbb{1}_{A_1} \circ T^k, \dots, \sum_{k=0}^{n-1} \mathbb{1}_{A_d} \circ T^k\right)$$

For $\alpha \in [0,1]$ and $\beta = (\beta_1, \ldots, \beta_d) \in [0,1]^d$ with $\sum_{i=1}^d \beta_i = 1$, we write $\zeta_{\alpha,\beta}$ for a $[0,1]^d$ -valued random variable whose distribution is characterized as follows:

- (1) If $0 < \alpha < 1$, the $\zeta_{\alpha,\beta}$ is equal in distribution to $(\xi_1, \ldots, \xi_d) / \sum_{i=1}^d \xi_i$, where ξ_1, \ldots, ξ_d are \mathbb{R}_+ -valued independent random variables with the one-sided α -stable distributions characterized by $\mathbb{E}[\exp(-\lambda\xi_i)] = \exp(-\beta_i\lambda^{\alpha}), \ \lambda > 0, \ i = 1, \ldots, d.$
- (2) If $\alpha = 1$, the $\zeta_{1,\beta}$ is equal a.s. to the constant β .
- (3) If $\alpha = 0$, the distribution of $\zeta_{0,\beta}$ is $\sum_{i=1}^{d} \beta_i \delta_{e^{(i)}}$ with $e^{(i)} = (1_{\{i=j\}})_{j=1}^{d} \in [0,1]^d$ for $i = 1, \ldots, d$.

The $\zeta_{\alpha,\beta}$ are called multidimensional generalized arcsine distributions, and appear as the limits of the joint distribution of the occupation time ratios of diffusions on multiray. See [1] and [5]. We now give our general limit theorem as follows.

Theorem 2.1. Under certain conditions, the following hold.

- (1) If S_n/n under $\nu' \stackrel{d}{\longrightarrow} \zeta$ as $n \to \infty$ for some probability measure $\nu' \ll \mu$, then $\zeta \stackrel{d}{=} \zeta_{\alpha,\beta}$ for some α and β , and S_n/n under $\nu \stackrel{d}{\longrightarrow} \zeta_{\alpha,\beta}$ as $n \to \infty$ for any probability measure $\nu \ll \mu$.
- (2) Let $\alpha \in [0,1)$ and $\beta_1, \ldots, \beta_d \neq 0$. Then the following are equivalent:
 - (i) S_n/n under $\nu \xrightarrow{d} \zeta_{\alpha,\beta}$ as $n \to \infty$ for any probability measure $\nu \ll \mu$.
 - (ii) There exists a regularly varying function R at ∞ with index $-\alpha$ such that

 $\mu(x \in Y; Tx, \ldots, T^n x \in A_i) \sim \beta_i R(n), \text{ as } n \to \infty, i = 1, \ldots, d.$

The case d = 2 was due to [4]. The proof in [4] was based on the moment method, which does not seem to be suitable for our multiray case. We adopt instead the double Laplace transform method, which was utilized in the study [1] of occupation times of diffusions on multiray. We will also explain applications to Lamperti's processes and interval maps with indifferent fixed points.

References

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