A FUNCTIONAL CENTRAL LIMIT THEOREM FOR NON-SYMMETRIC RANDOM WALKS ON STEP-r NILPOTENT COVERING GRAPHS

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A locally finite, connected and oriented graph $X = (V, E)$ is called a Γ-*nilpotent covering graph* if X is a covering graph of a finite graph $X_0 = (V_0, E_0)$ with a covering transformation group Γ which is a torsion free, finitely generated and nilpotent group of step $r\,(r \geq 2)$. Here V is a set of all vertices and E is a set of all oriented edges in X. For $e \in E$, the origin, the terminus and the inverse edge of e are denoted by $o(e), t(e)$ and $\overline{e}$, respectively. $E_x := \{e \in E \,|\, o(e) = x\}$ denotes the set of all edges whose origin is $x \in V$.

As is known, nilpotent covering graphs can be regarded as an extension of *crystal lattices* or *groups of polynomial growth* and the long time asymptotics for random walks (RWs) on them has been studied by several authors intensively and extensively (for instance, see [4, 1, 2]). We established a functional central limit theorem (FCLT; the Donsker–type invariance principle) for non-symmetric RWs on Γ-nilpotent covering graphs from a viewpoint of *discrete geometric analysis* developed by Sunada [5] under the assumption that Γ is free of step two (see [3]). However, we can completely relax the assumption and obtain an improved result on the FCLT. In this talk, we revisit this problem as a continuation of [3].

Let us consider a Γ-nilpotent covering graph $X = (V, E)$. We introduce a 1-step positive transition probability $p : E \longrightarrow (0, 1]$ which is invariant under Γ-actions. Then the transition probability p induces a RW $\{w_n\}_{n=0}^{\infty}$ with values in X. We may also consider the RW $\{\pi(w_n)\}_{n=0}^{\infty}$ on the quotient graph $X_0 = (V_0, E_0)$ by the Γ-invariance of p. Here $\pi : X \longrightarrow X_0$ is a covering map. Let $m : V_0 \longrightarrow (0, 1]$ be a normalized invariant measure on X_0 and we also write $m : V \longrightarrow (0, 1]$ for the Γ-invariant lift of m to X. Let $\mathrm{H}_1(X_0, \mathbb{R})$ and $\mathrm{H}^1(X_0, \mathbb{R})$ be the first homology group and the first cohomology group of X_0, respectively. In order to measure the homological drift of the RW, we define the *homological direction* of the RW on X_0 by $\gamma_p := \sum_{e \in E_0} p(e)m\big(o(e)\big)e \in \mathrm{H}_1(X_0, \mathbb{R})$. We call the RW on X_0 (m-)*symmetric* if $p(e)m\big(o(e)\big) = p(\overline{e})\big(t(e)\big)$ $(e \in E_0)$. It is clear that the RW is (m-)symmetric if and only if $\gamma_p = 0$.

By the celebrated theorem of Malćev, we find a connected and simply connected nilpotent Lie group G of step r such that Γ is isomorphic to the cocompact lattice in G. By virtue of the general theory of Lie algebras, its Lie algebra $\mathfrak{g}$ may have the direct sum decomposition $\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$ satisfying $[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j)} (i + j \leq r)$ and $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(i)}]\,(i = 1, \ldots, r-1)$. Now we take a canonical surjective linear map $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{R}) \longrightarrow \mathfrak{g}^{(1)}$ through the covering map π. By the discrete analogue of Hodge–Kodaira theorem (cf. [4]), an inner product

$$\langle\!\langle \omega, \eta \rangle\!\rangle_p := \sum_{e \in E_0} p(e)m\big(o(e)\big)\omega(e)\eta(e) - \langle \omega, \gamma_p \rangle \langle \eta, \gamma_p \rangle \quad \big(\omega, \eta \in \mathrm{H}^1(X_0, \mathbb{R})\big)$$

associated with the transition probability p is induced from the space of (modified) harmonic 1-forms on X_0 to $\mathrm{H}^1(X_0, \mathbb{R})$. Using the canonical map $\rho_{\mathbb{R}}$, we construct a flat metric g_0 called the *Albanese metric* on $\mathfrak{g}^{(1)}$ from the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_p$.

We consider a Γ-periodic realization $\Phi_0 : X \longrightarrow G$. In what follows, we take a reference point $x_* \in V$ such that $\Phi_0(x_*) = \mathbf{1}_G$. We call $\Phi_0 : X \longrightarrow G$ *modified harmonic* if

$$\sum_{e \in E_x} p(e) \log\Big(\Phi_0\big(o(e)\big)^{-1} \cdot \Phi_0\big(t(e)\big)\Big)\Big|_{\mathfrak{g}^{(1)}} = \rho_{\mathbb{R}}(\gamma_p) \quad (x \in V).$$

We note that such Φ_0 is uniquely determined, however, the modified harmonic realization has the ambiguity in the components corresponding to $\mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)}$. More precisely, $\log\big(\Phi_0(x)\big)\big|_{\mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(r)}}$ $(x \in V)$ is not be determined uniquely. The quantity $\rho_{\mathbb{R}}(\gamma_p)$ is called the *asymptotic direction*. We note that $\gamma_p = 0$ implies $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}_{\mathfrak{g}}$, however, the converse does not hold in general.

We introduce the family of *dilation operators* $\{\tau_\varepsilon\}_{\varepsilon \geq 0}$ acting on G. Let d_{CC} denote the Carnot–Carathéodory metric on G. Note that (G, d_{CC}) is not only a metric space but a geodesic space. Let $\mathcal{D}_n = \{k/n : k = 0, 1, \ldots, n\}$ be a $(1/n$-$)$partition of the time interval $[0,1]$ and set $\mathcal{Y}^{(n)}_{k/n} = \tau_{n^{-1/2}} \Phi_0(w_k)\,(n \in \mathbb{N},\, k = 0, 1, \ldots, n)$. Let $(\mathcal{Y}^{(n)}_t)_{0 \leq t \leq 1}$ be the G-valued continuous stochastic process given by the d_{CC}-geodesic interpolation of $\{\mathcal{Y}^{(n)}_{k/n}\}_{k=0}^n$. We write

$$H^1_{\mathbf{1}_G}([0,1], G) = \{h : [0,1] \to G \,|\, h : \text{absolutely continuous},\ h_0 = \mathbf{1}_G \text{ and } \|\dot{h}\|_{L^2} < \infty\}$$

for the usual Cameron–Martin subspace of the path space, where we are convinced $\dot{h}(t)$ belongs to the evaluation $\mathfrak{g}^{(1)}_{h(t)}$ and $\|\dot{h}\|_{L^2} := \int_0^1 \|\dot{h}(t)\|_{\mathfrak{g}^{(1)}}\, dt$. We denote by $\|\cdot\|_{\alpha\text{-Höl}}$ the α-Hölder norm with respect to d_{CC}. For every small parameter $\varepsilon > 0$, we set

$$\mathcal{W}^{1/2-\varepsilon}_{\mathbf{1}_G}([0,1], G) = \overline{H^1_{\mathbf{1}_G}([0,1], G)}^{\|\cdot\|_{(1/2-\varepsilon)\text{-Höl}}}.$$

Let $(Y_t)_{0 \leq t \leq 1}$ be the G-valued diffusion process which solves the SDE

$$dY_t = \sum_{i=1}^d V_i(Y_t) \circ dB^i_t + \beta(\Phi_0)(Y_t)\, dt, \quad Y_0 = \mathbf{1}_G,$$

where $\{V_1, \ldots, V_d\}$ be an orthonormal basis of $(\mathfrak{g}^{(1)}, g_0)$, the drift coefficient $\beta(\Phi_0) \in \mathfrak{g}^{(2)}$ is defined by

$$\beta(\Phi_0) := \sum_{e \in E_0} p(e) m\big(o(e)\big) \log\Big(\Phi_0\big(o(e)\big)^{-1} \cdot \Phi_0\big(t(e)\big)\Big)\Big|_{\mathfrak{g}^{(2)}},$$

and $(B_t)_{0 \leq t \leq 1} = (B^1_t, \ldots, B^d_t)_{0 \leq t \leq 1}$ is an $\mathbb{R}^d$-valued standard BM.

Then the refinement of the FCLT obtained in [3] is now stated as follows:

Theorem. (1) *Let $\Phi_0, \widetilde{\Phi}_0$ be two modified harmonic realizations. Then $\beta(\Phi_0) = \beta(\widetilde{\Phi}_0)$ holds.*

(2) *Under the assumption that $\rho_{\mathbb{R}}(\gamma_p) = \mathbf{0}_{\mathfrak{g}}$, we have, for every $\varepsilon > 0$,*

$$(\mathcal{Y}^{(n)}_t)_{0 \leq t \leq 1} \Longrightarrow (Y_t)_{0 \leq t \leq 1} \text{ in } \mathcal{W}^{1/2-\varepsilon}_{\mathbf{1}_G}([0,1]; G) \text{ as } n \to \infty.$$

References

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