Characterization of the explosion time for the Komatu–Loewner evolution

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1 Introduction

The Komatu–Loewner equation (K–L equation for short) is a correspondence to the Loewner equation in multiply connected domains. Bauer and Friedrich [1] established its concrete expression in standard slit domains of the upper half plane $\mathbb{H}$, and then, Chen, Fukushima et al. [2], [3], [4] investigated some properties of the Komatu–Loewner evolution generated by it.

In this talk, I will give a behavior of the image domain at the exlplosion time of this evolution, which is a refinement of a part of the study in [1]. The proof is based on a probabilistic expression of the solution that was developed in [2], [3] and [4], together with a general theory of complex analysis.

2 Notation and main results

We fix $N \in \mathbb{N}$. Let $C_j \subset \mathbb{H}$, $1 \leq j \leq N$, be slits parallel to the real axis and $K := \bigcup_{j=1}^{N} C_j$. We call a domain of the form $\mathbb{H} \setminus K$ a *standard slit domain*. The left and right end points of the slit C_j are denoted by $z_j = x_j + iy_j$ and $z_j^r = x_j^r + iy_j$ respectively. The slits $\{C_j; 1 \leq j \leq N\}$ are identified with a vector $\mathbf{s} = (y_1, \ldots, y_N, x_1, \ldots, x_N, x_1^r, \ldots, x_N^r) \in \mathbb{R}^{3N}$. We define the set $\mathcal{S}$ of all such "slit vectors" in $\mathbb{R}^{3N}$ as

$$\mathcal{S} := \{\mathbf{s} = (y_1, \ldots, y_N, x_1, \ldots, x_N, x_1^r, \ldots, x_N^r); y_j > 0, \text{ and } x_j < x_k^r \text{ or } x_k < x_j^r \text{ if } y_j = y_k \ (j \neq k)\}.$$

The slits and standard slit domain determined by $\mathbf{s} \in \mathcal{S}$ are denoted by $C_j(\mathbf{s})$, $1 \leq j \leq N$, and $D(\mathbf{s})$, respectively.

Take a standard slit domain $D = D(\mathbf{s})$ and a simple curve $\gamma : [0, t_\gamma) \to \overline{D}$ satisfying $\gamma(0) \in \partial\mathbb{H}$ and $\gamma(0, t_\gamma) \subset D$. For each $t \in [0, t_\gamma)$, there is a unique conformal map g_t from D onto another standard slit domain $D_t = D(\mathbf{s}(t))$ with the *hydrodynamic normalization*

$$g_t(z) = z + \frac{a_t}{z} + o(z^{-1}), \quad z \to \infty,$$

for some $a_t \geq 0$. The image $\xi(t) := g_t(\gamma(t))(= \lim_{z \to \gamma(t)} g_t(z)) \in \partial\mathbb{H}$ of the terminal point $\gamma(t)$ is called the *driving function* of g_t. The quantity a_t, called the *half plane capacity* of the set $\gamma[0, t]$ relative to g_t, is strictly increasing and continuous in t. Thus we can reparametrize the curve γ in such a way that $a_t = 2t$. Under these settings, $g_t(z)$ satisfies the following *K–L equation*:

$$\frac{d}{dt} g_t(z) = -2\pi \Psi_{\mathbf{s}(t)}(g_t(z), \xi(t)), \quad g_0(z) = z \in D. \tag{1}$$

The function $\Psi_{\mathbf{s}}(\cdot, w) = \Psi_D(\cdot, w)$, $w \in \partial\mathbb{H}$, is the conformal map from D onto some standard slit domain with w mapped to ∞, ∞ to 0 and $\Psi_D(z, w) \sim -\pi^{-1}(z - w)^{-1}$ as $z \to w$.

Since $\Psi_{\mathbb{H}}(z, w) = -\pi^{-1}(z - w)^{-1}$, the celebrated Loewner equation

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \xi(t)}, \quad g_0(z) = z \in \mathbb{H},$$

corresponds to the K–L equation in $D = \mathbb{H}$.

The end points $z_j(t) = x_j(t) + iy_j(t)$ and $z_j^r(t) = x_j^r(t) + iy_j(t)$ of the slits $C_{j,t} = C_j(\mathbf{s}(t))$ also satisfy the *K–L equation for slits*:

$$\begin{aligned}
\frac{d}{dt} y_j(t) &= -2\pi \Im \Psi_{\mathbf{s}(t)}(z_j(t), \xi(t)), \\
\frac{d}{dt} x_j(t) &= -2\pi \Re \Psi_{\mathbf{s}(t)}(z_j(t), \xi(t)), \\
\frac{d}{dt} x_j^r(t) &= -2\pi \Re \Psi_{\mathbf{s}(t)}(z_j^r(t), \xi(t)).
\end{aligned} \tag{2}$$

We now follow this procedure in the opposite direction. Namely, for a given driving function $\xi \in C([0,\infty);\mathbb{R})$, we first solve the K–L equation (2) for slits $\mathbf{s}(t)$ and then solve (1) for $g_t(z)$, $z \in D$. We denote by t_ξ the explosion time for the ODEs (2) and put $F_t := \{z \in D; t_z \le t\}$, $t < t_\xi$, where

$$t_z = t_\xi \wedge \sup\{t > 0; |g_t(z) - \xi(t)| > 0\}, \quad z \in D,$$

is the explosion time of $g_t(z)$. It is possible to check that g_t, $t \in [0, t_\xi)$, is a unique conformal map from $D \setminus F_t$ onto $D_t = D(\mathbf{s}(t))$ satisfying the hydrodynamic normalization with $a_t = 2t$. The bounded set F_t is not necessarily a curve but a (compact) $\mathbb{H}$-*hull* in the sense that $F_t = \mathbb{H} \cap \overline{F_t}$ and that $\mathbb{H} \setminus F_t$ is simply connected. We call both g_t and F_t the *Komatu–Loewner evolution driven by* $\xi(t)$.

It is a natural problem what happens if t_ξ is finite. A reasonable guess is that the evolution F_t should hit the slits $\bigcup_j C_j$ at the time t_ξ. In terms of the slits $C_{j,t} = C_j(\mathbf{s}(t))$ of D_t, this means that $C_{j,t}$ is absorbed into the real axis for some j as claimed in [1, Theorem 4.1]. Justifying this description is, however, not trivial because the solution to (2) belongs to the space of slits $\mathcal{S}$, not $\mathbb{R}^{3N}$, and the slits $C_{j,t}$ may degenerate to one point or collide with each other before reaching $\partial\mathbb{H}$.

Our main theorem justifies the above description in the following manner:

Theorem 1. *Let* $R(w, \mathbf{s}) := \min_{1 \le j \le N} \operatorname{dist}(C_j(\mathbf{s}), w)$ *for* $w \in \partial\mathbb{H}$ *and* $\mathbf{s} \in \mathcal{S}$*. If* $t_\xi < \infty$*, then it holds that* $\lim_{t \nearrow t_\xi} R(\xi(t), \mathbf{s}(t)) = 0$.

For the proof, it suffices to extend the solution $\mathbf{s}(t)$ beyond t_ξ if the conclusion does not hold. To this end, we interpret the complicated evolution g_t and F_t in D as a simpler one g_t^0 and F_t in $\mathbb{H}$ by "forgetting the slits," the technique employed in [4]. g_t^0 and F_t extend to a Loewner evolution in $\mathbb{H}$ over the time interval $[0, t_\xi]$. Then, by a version of Carathéodory's kernel theorem (cf. [5, Theorem 15.4.7]), $\{g_t \circ (g_t^0)^{-1}; t < t_\xi\}$ extends to a family of conformal maps over $[0, t_\xi]$. This implies that the limit $\mathbf{s}(t_\xi) = \lim_{t \nearrow t_\xi} \mathbf{s}(t)$ still represents N slits in $\mathbb{H}$, which is a contradiction.

If time permits, I will provide an example where the explosion time for the *stochastic Komatu–Loewner evolution (SKLE)* is finite with probability one. We define the *domain constant* k as

$$k(w, \mathbf{s}) := 2\pi \lim_{z \to w} \left(\Psi_{\mathbf{s}}(z, w) + \frac{1}{\pi} \frac{1}{z - w} \right), \quad w \in \partial\mathbb{H}, \mathbf{s} \in \mathcal{S}.$$

$\mathrm{SKLE}_{\sqrt{6},k}$ is a K–L evolution driven by the random function ξ determined by the system of SDEs (2) and

$$d\xi(t) = -k(\xi(t), \mathbf{s}(t))dt + \sqrt{6}dB_t \tag{3}$$

where B_t is a one-dimensional standard Brownian motion. Then we have the following:

Proposition 2. *Let* ζ *be the explosion time for the SDEs* (2) *and* (3)*. It holds that* $\zeta < \infty$ *almost surely.*

This is proven by interpreting SLE_6 as $\mathrm{SKLE}_{\sqrt{6},k}$, i.e., "recalling the slits," which is an idea in the opposite direction to the proof of Theorem 1.

References

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