The invariant measure and the flow associated to the Φ^4_3-quantum field model

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We consider the invariant measure and flow for the stochastic quantization equation associated to the Φ^4_3-model on the torus, which appears in quantum field theory. By virtue of Hairer's breakthrough, such nonlinear stochastic partial differential equations became solvable and are intensively studied now. In this talk, we present a direct construction to both a global solution and invariant measure for this equation.

Let $m_0 > 0$, Λ be the 3-dimensional torus i.e. $\Lambda = (\mathbb{R}/\mathbb{Z})^3$, and μ_0 be the centered Gaussian measure on the space of Schwartz distributions $\mathcal{S}'(\Lambda)$ with the covariance operator $[2(-\triangle + m_0^2)]^{-1}$. We remark that μ_0 is different from the Nelson's Euclidean free field measure by the scaling $\sqrt{2}$. In order to adjust our setting to those of known results, we define μ_0 as above. In the constructive quantum field theory, there was a problem to construct a measure

$$\mu(d\phi) = Z^{-1} \exp\left(-U(\phi)\right) \mu_0(d\phi)$$

where

$$U(\phi) = \int_\Lambda \left(\frac{1}{4}\phi(x)^4 - C\phi(x)^2 \right) dx$$

and Z is the normalizing constant. Since the support of μ_0 is in the space of tempered distributions, ϕ^4 and ϕ^2 are not defined in usual sense. So, we approximate ϕ and take the limit.

Let $\langle f, g\rangle$ be the inner product on $L^2(\Lambda;\mathbb{C})$. For $k \in \mathbb{Z}^d$, define $e_k(x) := e^{2\pi \mathrm{i} k\cdot x}$ where $k \cdot x := k_1x_1 + k_2x_2 + k_3x_3$. For $N \in \mathbb{N}$, denote $\{j \in \mathbb{Z}; |j| \leq N\}$ by $\mathbb{Z}_N$, and let P_N be the mapping from $\mathcal{S}'(\Lambda)$ to $L^2(\Lambda;\mathbb{C})$ given by

$$P_N\phi := \sum_{k\in\mathbb{Z}_N^3} \langle \phi, e_k\rangle e_k.$$

Define a function U_N on $\mathcal{S}'(\Lambda)$ by

$$U_N(\phi) = \int_\Lambda \left\{ \frac{1}{4}(P_N\phi)(x)^4 - \frac{3}{2}\left(C_1^{(N)} - 3C_2^{(N)}\right)(P_N\phi)(x)^2 \right\} dx$$

where

$$C_1^{(N)} = \frac{1}{2}\sum_{k\in\mathbb{Z}_N^3} \frac{1}{k^2+m_0^2}, \quad C_2^{(N)} = \frac{1}{2}\sum_{l_1,l_2\in\mathbb{Z}_N^3} \frac{1}{(l_1^2+m_0^2)(l_2^2+m_0^2)(l_1^2+l_2^2+(l_1+l_2)^2+3m_0^2)}.$$

We remark that $\lim_{N\to\infty} C_1^{(N)} = \lim_{N\to\infty} C_2^{(N)} = \infty$, and $C_1^{(N)}$ and $C_2^{(N)}$ are called renormalization constants. Consider the probability measure μ_N on $\mathcal{S}'(\Lambda)$ given by

$$\mu_N(d\phi) = Z_N^{-1} \exp\left(-U_N(\phi)\right) \mu_0(d\phi)$$

where Z_N is the normalizing constant. We remark that $\{\mu_N\}$ is the approximation sequence of the Φ^4_3-measure which will be constructed below as the invariant measure of the associated flow.

Now we consider the stochastic quantization equation associated to $\{\mu_N\}$ as follows.

$$\begin{cases} d\tilde{X}^N_t(x) &= dW_t(x) - (-\triangle + m_0^2)\tilde{X}^N_t(x)dt \\ & \qquad - \left\{ P_N[(P_N\tilde{X}^N_t)^3](x) - 3\left(C_1^{(N)} - 3C_2^{(N)}\right) P_N\tilde{X}^N_t(x) \right\} dt \\ \tilde{X}^N_0(x) &= \xi_N(x) \end{cases}$$

where $W_t(x)$ is a white noise with parameter $(t,x) \in [0,\infty) \times \Lambda$ and $\xi_N(x)$ is a random variable which has μ_N as the law, and independent of W_t. We remark that μ_N is the invariant measure with respect to the semigroup generated by the solution to the equation. Let $X^N := P_N \tilde{X}^N$ for $N \in \mathbb{N}$. Then, X^N satisfies the stochastic partial differential equation

$$\begin{cases} dX^N_t(x) &= P_N dW_t(x) - (-\triangle + m_0^2)X^N_t(x)dt \\ & \qquad - \left\{ P_N[(X^N_t)^3](x) - 3(C_1^{(N)} - 3C_2^{(N)})X^N_t(x) \right\} dt \\ X^N_0(x) &= P_N\xi_N(x) \end{cases} \tag{1}$$

To apply the Hairer's reconstruction method, which enables us to transform (1) for a solvable partial differential equation, we supplementary introduce the infinite-dimensional Ornstein-Uhlembeck process Z as follows. Let Z be the solution to the stochastic partial differential equation on Λ

$$\begin{cases} dZ_t(x) &= dW_t(x) - (-\triangle + m_0^2)Z_t(x)dt, \quad (t,x) \in [0,\infty) \times \Lambda \\ Z_0(x) &= \zeta(x), \qquad\qquad\qquad\qquad\qquad\quad x \in \Lambda \end{cases}$$

where ζ is a random variable which has μ_0 as its law and is independent of W_t and ξ_N. Let $X^{N,(2)}_t := X^N_t - \mathscr{Z}^{(1,N)}_t + \mathscr{Z}^{(0,3,N)}_t$ for $t \in [0,\infty)$ where

$$\mathscr{Z}^{(0,3,N)}_t := \int_0^t e^{(t-s)(\triangle - m_0^2)} \left(P_N(P_N Z_s)^3 - 3C_1^N P_N Z_s \right) ds, \quad t \in [0,\infty),$$

and decompose $X^{N,(2)}$ into $X^{N,(2),<}$ and $X^{N,(2),\geqslant}$ by means of paraproduct. Then, we have a solvable, coupled, semilinear and dissipative parabolic partial differential equation, which the pair $(X^{N,(2),<}, X^{N,(2),\geqslant})$ satisfies. By applying the technique of the semilinear and dissipative parabolic equation, we obtain some estimates for $X^{N,(2),<}$ and $X^{N,(2),\geqslant}$, which yields the tightness of $X^{N,(2)}$. As the result we obtain the following theorem for the Φ^4_3-measure and the associated flow.

Theorem 1. *For $\varepsilon \in (0,1]$ sufficiently small, $\{X^N\}$ is tight on $C([0,\infty); B^{-1/2-\varepsilon}_{4/3})$, where $B^s_{p,r}$ is the Besov space. Moreover, if X is a limit of a subsequence $\{X^{N(k)}\}$ of $\{X^N\}$ on $C([0,\infty); B^{-1/2-\varepsilon}_{4/3})$, then X is a continuous Markov process on $B^{-1/2-\varepsilon}_{4/3}$, the limit measure μ of the associated subsequence $\{\mu_{N(k)}\}$ is an invariant measure with respect to X and it holds that*

$$\int \|\phi\|^4_{B_4^{-1/2-\varepsilon}} \mu(d\phi) < \infty.$$