Heat trace asymptotics for equiregular sub-Riemannian manifolds

Yuzuru INAHAMA *

This is a jointwork with Setsuo TANIGUCHI (Kyushu University) and can be found at arXiv Preprint Server (arXiv:1706.02450).

We study a "div-grad type" sub-Laplacian with respect to a smooth measure and its associated heat semigroup on a compact equiregular sub-Riemannian manifold. We prove a short time asymptotic expansion of the heat trace up to any order. Our main result holds true for any smooth measure on the manifold, but it has a spectral geometric meaning when Popp's measure is considered. Our proof is probabilistic. In particular, we use S. Watanabe's distributional Malliavin calculus.

In Introduction of his textbook on sub-Riemannian geometry [3], R. Montgomery emphasized the importance of spectral geometric problems in sub-Riemannian geometry by asking "Can you 'hear' the sub-Riemannian metric from the spectrum of its sublaplacian?" (Of course, this is a slight modification of M. Kac's renowned question.) In the same paragraph, he also mentioned Malliavin calculus, which is a powerful infinite-dimensional functional analytic method for studying stochastic differential equations (SDEs) under the Hörmander condition on the coefficient vector fields.

However, there is no canonical choice of measure on a general sub-Riemannian manifold and hence no canonical choice of sub-Laplacian. Therefore, in order to pose spectral geometric questions, one should consider a subclass of sub-Riemannian manifolds. In this regard, the class of equiregular sub-Riemannian manifolds seems suitable for the following reason. As Montgomery himself proved in Section 10.6, [3], there exists a canonical smooth volume called Popp's measure on an equiregular sub-Riemannian manifold. Popp's measure is determined by the sub-Riemannian metric only.

In this talk we prove a short time asymptotic expansion of the heat trace up to an arbitrary order on a compact equiregular sub-Riemannian manifold. Our main tool is Watanabe's distributional Malliavin calculus.

Let $M = (M, \mathcal{D}, g)$ be a sub-Riemannian manifold and μ be a smooth volume on M. ($\mathcal{D}$ is a subbundle of TM that satisfies the Hörmander condiyion at every point and g is an inner product on $\mathcal{D}$.) We study the second-order differential operator of the form $\triangle = \mathrm{div}_\mu \nabla^{\mathcal{D}}$, where $\nabla^{\mathcal{D}}$ is the horizontal gradient in the direction of $\mathcal{D}$ and div_μ is the divergence with respect to μ. (In our convention, $\triangle$ is a non-positive operator.) By the way it is defined, $\triangle$ with its domain being $C_0^\infty(M)$ is clearly symmetric on $L^2(\mu)$. If M is compact, then $\triangle$ is known to be essentially self-adjoint on $C^\infty(M)$ and $e^{t\triangle/2}$ is of trace class for every $t > 0$, where $(e^{t\triangle/2})_{t\geq 0}$ is the heat semigroup associated with $\triangle/2$.

Now we are in a position to state our main result in this paper. As we have already mentioned, it has a spectral geometric meaning when μ is Popp's measure.

*Kyushu University, Japan. Email:inahama@math.kyushu-u.ac.jp

Theorem 1 *Let M be a compact equiregular sub-Riemannian manifold of Hausdorff dimension ν and let μ be a smooth volume on M. Then, we have the following asymptotic expansion of the heat trace:*

$$\mathrm{Trace}(e^{t\triangle/2}) \sim \frac{1}{t^{\nu/2}}(c_0 + c_1 t + c_2 t^2 + \cdots) \qquad \text{as } t \searrow 0 \tag{1}$$

for certain constants $c_0 > 0$ and $c_1, c_2, \ldots \in \mathbb{R}$.

Since the asymptotic expansion in Theorem 1 is up to an arbitrary order, we can prove meromorphic prolongation of the spectral zeta function associated with $\triangle$ by a standard argument. Denote by $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ be all the eigenvalues of $-\triangle$ in increasing order with the multiplicities being counted and set

$$\zeta_\triangle(s) = \sum_{i=0}^{\infty} \lambda_i^{-s} \qquad (s \in \mathbb{C},\ \Re s > \frac{\nu}{2}).$$

By the Tauberian theorem, the series on the right hand side absolutely converges and defines a holomorphic function on $\{s \in \mathbb{C} \mid \Re s > \nu/2\}$.

Corollary 2 *Let assumptions be the same as in Theorem 1. Then, $\zeta_\triangle$ admits a meromorphic prolongation to the whole complex plane $\mathbb{C}$.*

To the best of our knowledge, Theorem 1 and Corollary 2 seem new for a general compact equiregular sub-Riemannian manifold. It should be noted, however, that the leading term of the asymptotics (1) is already known. See Métivier (1976) for example. No explicit value of c_0 is known in general. For some concrete examples or relatively small classes of compact equiregular sub-Riemannian manifolds, the full asymptotic expansion (1) or the meromorphic extension of the spectral zeta function was proved. Most of such classes are subclasses of step-two or corank-one sub-Riemannian manifolds.

Our proof of Theorem 1 is based on Takanobu's beautiful result [2] on the short time asymptotic expansion of hypoelliptic heat kernels on $\mathbb{R}^d$ on the diagonal. Using results in Taniguchi (1983) and Grong-Thalmaier (2016)/Thalmaier (2016), we can do the same thing on a compact manifold. (The former developed manifold-valued Malliavin calculus under the partial Hörmander condition, while the latter constructed $\triangle/2$-diffusion process on M via stochastic parallel transport.)

References

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