Renormalized Gibbs measures and applications to QFT

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This is a joint work with Oliver Matte in Aalborg university [5]. The Nelson Hamiltonian with UV(ultraviolet) cutoff parameter $\varepsilon > 0$ is given by

$$H_\varepsilon = H_{\rm p} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f} + g\phi(\varrho_\varepsilon(\cdot - x)).$$

Here $H_{\rm p} = -\frac{1}{2}\Delta + V(x)$ is a Schrödinger operator. The operator $H_{\rm f} = d\Gamma(\omega(-i\nabla))$ is the free field Hamiltonian and $\phi(\varrho_\varepsilon(\cdot - x))$ is a Gaussian random variable with cutoff function given by

$$\hat{\varrho}_\varepsilon(k) = \frac{e^{-\varepsilon|k|^2/2}}{\sqrt{\omega(k)}} \mathbb{1}_{|k|>\lambda} \in L^2(\mathbb{R}^3) \quad \varepsilon > 0.$$

E. Nelson [7] introduces the renormalization term:

$$E_\varepsilon = -\frac{1}{2}\int_{|k|>\lambda} \frac{|\hat{\varrho}_\varepsilon(k)|^2}{\omega(k) + |k|^2/2} dk.$$

Here we noice that $E_\varepsilon \to -\infty$ as $\varepsilon \downarrow 0$. It is shown in [8, 7, 1] that there exists a self-adjoint operator $H_{\rm ren}$ such that

$$\lim_{\varepsilon\downarrow 0} e^{-T(H_\varepsilon - g^2 E_\varepsilon)} = e^{-TH_{\rm ren}}.$$

The important fact is that we can *not* see the explicit form of $H_{\rm ren}$, it is however shown in [3] that $H_{\rm ren}$ has the ground state, and it is unique by [6]. Let $\Psi_{\rm g}$ be the ground state of $H_{\rm ren}$. Let $0 \le \phi \in L^2(\mathbb{R}^3)$ and since $(\phi \otimes \mathbb{1}, \Psi_{\rm g}) \neq 0$, we have

$$(\Psi_{\rm g}, O\Psi_{\rm g}) = \lim_{T\to\infty} \lim_{\varepsilon\downarrow 0} \frac{(e^{-TH_\varepsilon}\phi \otimes \mathbb{1}, Oe^{-TH_\varepsilon}\phi \otimes \mathbb{1})}{\|e^{-TH_\varepsilon}\phi \otimes \mathbb{1}\|^2}$$

for any bounded operators O. On the Wiener space $(\Omega, \mathcal{F}, W)$ the finite volume Gibbs measure is defined by

$$\mu_T(A) = \frac{1}{Z_T}\int_{\mathbb{R}^3} dx \mathbb{E}_W^x \left[\mathbb{1}_A \phi(B_{-T})\phi(B_T) e^{\frac{g^2}{2}S_{\rm ren}}\right]$$

for $A \in \mathcal{F}$, where $(B_t)_{t\in\mathbb{R}}$ is BM on the whole line. For some O it follows that $(\Psi_{\rm g}, O\Psi_{\rm g}) = \lim_{T\to\infty} \mathbb{E}_{\mu_T}[O_T]$ with some integrant O_T. Let $\mathcal{F}_{[-S,S]} = \sigma(B_r, r \in [-S, S])$ and we set $\mathcal{G} = \sigma(\cup_{S\ge 0}\mathcal{F}_{[-S,S]})$.

Theorem 0.1 *[4, 2, 5] There exists a probability measure μ_∞ on $(\Omega, \mathcal{G})$ such that $\mu_T \to \mu_\infty$ as $T \to \infty$ in the local weak sense. I.e., $\mu_T(A) \to \mu_\infty(A)$ for $A \in \mathcal{F}_{[-S,S]}$ for arbitrary S.*

We show several applications in terms of infinite volume Gibbs measure μ_∞ in [5].

(1.Super exponential decay of the number of bosons) Let N_Λ be the truncated number operator. Then

$$(\Psi_{\rm g}, e^{+\beta N_\Lambda}\Psi_{\rm g}) = \mathbb{E}_{\mu_\infty}[e^{-(1-e^{+\beta})\int_{-\infty}^0 ds\int_0^\infty dt W_\Lambda}] < \infty$$

for all $\beta \geq 0$.

(2. Gaussian decay) It follows that

$$(\Psi_{\rm g}, e^{+\beta\phi(f)^2}\Psi_{\rm g}) = \frac{1}{\sqrt{1-\beta\|f\|^2/2}}\mathbb{E}_{\mu_\infty}\left[e^{+\frac{\beta S_\infty^2}{2(1-\beta\|f\|^2/2)}}\right].$$

In particular $(\Psi_{\rm g}, e^{\beta\phi(f)^2}\Psi_{\rm g}) < \infty$ for $\beta < 1/(2\|f\|^2)$ and $\lim_{\beta\uparrow 1/(2\|f\|^2)}(\Psi_{\rm g}, e^{\beta\phi(f)^2}\Psi_{\rm g}) = \infty$.

(3. Spatial decay) We have by [6]

$$\Psi_{\rm g} = e^{-T(H_{\rm ren}-E)}\Psi_{\rm g} = e^{TE}e^{-\int_0^T V(B_s+x)ds}e^{\frac{g^2}{2}S_{\rm ren}}e^{a^*(U_T)}e^{-TH_{\rm f}}e^{a(\bar{U}_T)}\Psi_{\rm g}(B_t+x).$$

Here $U_T = -\frac{g}{\sqrt{2}}\int_0^T \frac{e^{-|s|\omega(k)}}{\sqrt{\omega(k)}}e^{-ikB_s}ds$ and $E = \inf\sigma(H_{\rm ren})$. Under some condition on V it follows that

$$\|\Psi_{\rm g}(x)\| \leq Ce^{-c|x|}$$

for a.e.$x \in \mathbb{R}^3$.

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