Arbitrage theory in large financial markets

Yushi Hamaguchi* Kyoto university

1 Introduction

In mathematical finance classical market models consist of an $\mathbb{R}^d$-valued semimartingale on some probability space which describes the discounted price process of d financial assets. In this talk we consider a large financial market which consists of infinitely many financial assets. This concept was introduced by Y. Kabanov and D. Kramkov [1] to formalize a market where hundreds of financial assets are available and then several notions and characterizations of arbitrage in large markets were developed [2].

An arbitrage opportunity is the possibility to make a profit in a financial market without risk. The principle of no-arbitrage states that a mathematical model of a financial market should not allow for arbitrage opportunities. The condition of no-arbitrage is essentially equivalent to the existence of an equivalent martingale measure for the price process and this is crucial to the modern theory of finance such as the option pricing theory or the utilty maximization problem.

2 Generalized strategies and arbitrage

We consider a large financial market model consisting of a sequence of semimartingales $\mathbb{S} = \{(S_t^n)_{t\in[0,T]}\}_{n\in\mathbb{N}}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ which describes the discounted price process of infinitely many financial assets. We denote the n-th small market by $\mathbb{S}^n = (S^k)_{k\le n}$.

Definition 1.

- *A strategy in the n-th small market is an $\mathbb{R}^n$-valued predictable process which is $\mathbb{S}^n$-integrable. The wealth process corresponding to a strategy H is the stochastic integral $H \bullet \mathbb{S}^n$.*
- *We say that a strategy H in the n-th small market is admissible if its wealth process $H \bullet \mathbb{S}^n$ is bounded from below.*
- *We denote the set of all admissible strategies in the n-th small market by $\mathcal{H}^n$ and the set of attainable claims in the n-th small market by $\mathcal{K}^n = \{(H \bullet \mathbb{S}^n)_T | H \in \mathcal{H}^n\}$.*
- *We say that the n-th small market satisfies NA (No Arbitrage) if $\mathcal{K}^n \cap L_+^0 = \{0\}$ where L_+^0 denotes the convex cone of nonnegative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.*

*hamaguchi@math.kyoto-u.ac.jp

The admissibility condition is imposed to exclude so called "doubling strategies". The condition NA says that there exists no admissible strategy which satisfies $(H \bullet \mathbb{S}^n)_T \geq 0$ *a.s.* (without risk) and $\mathbb{P}\{(H \bullet \mathbb{S}^n)_T > 0\} > 0$ (positive profit).

The corresponding notions of trading strategies or arbitrage in a large financial market $\mathbb{S}$ are as follows;

Definition 2.

- *For each $n \in \mathbb{N}$, let H^n be an $\mathbb{R}^n$-valued, predictable, $\mathbb{S}^n$-integrable process. A sequence $\mathbb{H} = (H^n)_{n\in\mathbb{N}}$ is called generalized strategy if $(H^n \bullet \mathbb{S}^n)$ converges in the Emery topology to a semimartingale Z, which is called a generalized stochastic integral (or a generalized wealth process) and denoted by $Z = \mathbb{H} \bullet \mathbb{S}$.*
- *A generalized strategy $\mathbb{H} = (H^n)_{n\in\mathbb{N}}$ is called admissible if the approximating sequence $(H^n)_{n\in\mathbb{N}}$ is uniformly admissible.*
- *We denote the set of all generalized admissible strategies by $\mathcal{H}$ and the set of approximately attainable claims in the large market by $\mathcal{K} = \{(\mathbb{H} \bullet \mathbb{S})_T | \mathbb{H} \in \mathcal{H}\}$.*
- *We say that a large market satisfies NGA (No Generalized Arbitrage) if $\mathcal{K} \cap L^0_+ = \{0\}$.*

The notion of generalized stochastic integral with respect to a sequence of semimartingales was introduced by De. Donno and Pratelli [3], which formalizes the idea of a trading strategy in which each asset can contribute, possibly with an infinitesimal weight.

3 A change of numéraire

We deal with the change of numéraire problem in large financial markets. Consider a model $\mathbb{X} = ((S^n)_{n\in\mathbb{N}}, 1, V)$, where V is a positive semimartingale describing a new numéraire (that is, a new currency unit). If the currency unit is changed to the new numéraire V, the price process $\mathbb{X}$ will be multiplied by the exchange ratio $\frac{1}{V}$ and the price process under this new numéraire becomes $\mathbb{Z} = ((\frac{S^n}{V})_{n\in\mathbb{N}}, \frac{1}{V}, 1)$. We will talk about the condition under which the NGA condition is preserved under a change of numéraire.

References

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